Lecture Notes from the Summer School of DFG SPP1257 Global Water Cycle

September 12-16, 2011
Ed. by A. Eicker, J. Kusche
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Summerschool ‘Global Hydrological Cycle’ of the DFG-SPP1257

Mayschoss/Ahr, September 12-16, 2011

In 2006, the German Research Association DFG had established the coordinated Priority Program SPP1257 Mass distribution and Mass Transport in the Earth System. According to DFG’s philosophy, SPP’s are meant to enable broad-scale research in new, emerging fields. The objective of the SPP1257 was to facilitate integrated analysis of novel-type data collected from dedicated gravity field and radar altimetry satellite missions, to improve our knowledge about mass distribution and mass transport processes within the Earth system such as melting of continental ice sheets and glaciers, changes in ocean circulation pattern and in sea level, variations of surface and ground water levels and river discharge, glacial-isostatic adjustment, mantle convection and tectonics, and to investigate interactions between these processes. During six years, many Ph.D. students and postdocs from more than 30 institutions worked together in collaborative projects.

These lecture notes were compiled on the occasion of the summer school Global Hydrological Cycle, organized by the SPP1257 at September 12-16, 2011 in Mayschoss/Ahr, in which about 70 Ph.D. students, postdoc researchers and master students participated. The challenge imposed on the lecturers was to familiarize students with widely different background (geodesy, hydrology, oceanography, geophysics, mathematics) with

- concepts of observation systems and data processing, such as analysis of data from the Gravity Recovery and Climate Experiment (GRACE) gravity mission, and from radar altimetric satellite missions, associated problems such as noise, spatio-temporal sampling and aliasing, data post-processing techniques such as spherical harmonic synthesis and analysis, gridding, smoothing, covariance analysis, EOF analysis, and

- concepts of modelling and interpretation in hydrology and hydro-meteorology, oceanography and sea level, tides, ice sheet modelling, climate dynamics, and solid-Earth geodynamics.

The focus of the summer school was on concepts, and technical proofs were avoided. Lectures were accompanied by exercises, practicals and group work. Last not least, exciting discussions could be continued during barbecue, walks on the Rotweinwanderweg, and in the cellar of the Winzergenossenschaft.

Thanks go to all participants and, in particular, to the lecturers of the summer school, who decided to make almost all lecture material, data sets and codes freely available.

Jürgen Kusche and Annette Eicker

Bonn, February 11, 2013
Mass transport and mass distribution in the system Earth
Outline

- Brief overview on the continental water cycle
- In situ data
- Global land surface models
- Satellite data for hydrology
- Monitoring surface waters by radar altimetry
- Altimetry-derived water level data bases
- Validation; derived products
- Soil moisture from space
- Space gravimetry (GRACE)
- problems with current missions
- Future prospects
Causes of spatio-temporal change of the continental water cycle

- Climate variability (natural and anthropogenic)

- Direct human effects:
  - groundwater mining
  - irrigation
  - dam building
  - urbanization
  - deforestation
  - change in land use
Continental Water Cycle

Water flux exchanges between reservoirs

- Water mass exchanges?
- Time scales of exchanges?
- Water holding capacity of reservoirs?
- Rates of water renewal inside reservoirs

Mechanisms

- Mass and energy transfert between land surface and lower atmosphere
- Lower atmosphere dynamics
- Biogeochemical processes
- etc.

Applications

- Weather forecast
- Climate modelling
- Water resources management
- Natural Hazards:
  - floods, droughts
- Agriculture (irrigation)
- Hydro-electric energy production
- Fluvial navigation
- Land use and management
- Carbon cycle
- Sediment transport
- Sea level change
- Etc.
Importance of each parameter

- Precipitation:
  - Main forcing term
  - Controls water storage and runoff

- Evaporation; Vegetation transpiration:
  - Drives mass and energy exchange between lower atmosphere and surface
  - Depends on solar radiation, lower atmospheric state, soil wetness, type of vegetation
  - Direct link with air temperature

- Runoff:
  - Linked to basin-scale water budget

Water balance at river basin scale
\[
\frac{dW}{dt} = P - E - R
\]

Total water storage: importance of each component:

- Surface waters:
  - Direct response to precipitation
  - Flood plains control ecosystem dynamics and \( CO_2 \) exchanges with the atmosphere

- Soil moisture:
  - Drives evapotranspiration and vegetation growth (\( \rightarrow \) carbon cycle)
  - Important parameter for weather forecast and climate modelling

- Snow pack:
  - Influence planet albedo, hence total radiative budget

- Ground waters:
  - Main water resource in semi arid regions
Water Balance Equation
(river basin scale)

\[ \frac{dW}{dt} = P - E - R \]

W: Land water mass (surface and underground waters; snowpack)

P: Precipitation

E: Evapotranspiration

R: Runoff

Approach

Observations: in situ and remote sensing

Modelling

Data assimilation
Global Runoff Data Center

Water level and discharge data available in the GRDC data base (status in March 2011)

Global Soil Moisture Data Bank
• Compute mass and energy budget at the atmosphere-soil interface + water storage in the different reservoirs + runoff

• Input parameters: low atmospheric state (T, H, wind speed) + mass and energy fluxes (precipitation and radiation)
<table>
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<tr>
<th>Remote sensing technique</th>
<th>Soil moisture</th>
<th>Ground waters</th>
<th>Snow pack</th>
<th>Surface waters (extent, level, volume, discharge)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visible Imagery</td>
<td>Extent</td>
<td>Extent</td>
<td>Extent</td>
<td>Extent</td>
</tr>
<tr>
<td>Passive and active microwaves (Radiometry)</td>
<td>Extent</td>
<td>Volume</td>
<td>Extent Thickness</td>
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<td>Altimetry</td>
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<tr>
<td>Space Gravimetry (GRACE)</td>
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<td>Total water mass</td>
</tr>
</tbody>
</table>

- Surface water level by satellite altimetry
- Soil moisture
- Total water volume stored in the water column
- GRACE space gravimetry
Primary and derived hydrological products
(by combining obs. from different remote sensing techniques)

- Soil moisture: microwaves + SMOS
- Water levels: altimetry Topex/Poseidon, Jason-1/2, ERS, Envisat
- Snow pack: microwaves, GRACE
- Land water storage: GRACE
- Surface water volume: imagery + altimetry
- River discharge: altimetry + modelling (Manning equation)
- Ground waters: GRACE + altimetry + imagery + SMOS
- Evapotranspiration (basin-scale): GRACE + runoff + precipitation

Satellite altimetry

Space gravimetry
Satellite altimetry to measure surface water levels
Satellite altimetry

Global Earth coverage in a few days
Sea level measurements by satellite altimetry

Satellite Altimetry:

Topex/Poseidon (1992-2006)
Jason-1 (2001-)
ENVISAT (2002-)
GFO (2000-)
Jason-2 (2008-)
Important achievements in oceanography with high-precision satellite altimetry

Turbulent ocean

Ocean tides

El Nino

Sea level change

Altimetry-derived global mean sea level (1993-2011)

Rate of sea level rise: 3.3 +/- 0.4 mm/yr
Examples of altimetric coverage over lakes

- **Athabasca Lake**
- **Baikal Lake**
- **Lake Victoria (Afrique de l’Est)**
- **Lake Tharthar (Irak)**

**Water level variations**

- **Lac Victoria**
  - lat=1.00
  - lon=35.00
- **Lac Tharthar**
  - lat=34.00
  - lon=43.20

**Hydroweb (Legos)**
Lake Victoria

Altimetry-based lake level

1992 2009

Lake Tanganika

Lake Malawi

East African Lakes

Lake Poyang, China

Lake Nasser, Nile basin

HYDROWEB, Legos
Example of 'virtual' station

Example of altimetric coverage over rivers
‘Altimetry’ virtual stations supplementary of in situ gauges

Red: Envisat stations
Yellow: In situ gauges (ANA, Brazil)

Validation of altimetric water levels
Comparison of altimetry and in situ water levels

Comparaison des hauteurs T/P254 et des hauteurs à Barranco Branco (d = 7 km)

72
73
74
75
76
77
78
79
80
81
82


Hauteur d'eau (m)

Différence de hauteur (m)

RMS global : 0.18 m
RMS hautes eaux : 0.17 m
RMS basses eaux : 0.19 m

Variations temporelles des hauteurs T/P 094 à proximité de Bezdan (d = 20 km)

78
79
80
81
82
83
84
85
86
87
88


Hauteur d'eau (m)

Différence de hauteur (m)

Hauteurs T/P 094
Hauteurs reconsitutées à Bezdan
Ecarts entre les deux séries

RMS global = 0.37

Comparaison des hauteurs T/P076 et des hauteurs à Jalauaca (d=48km)

15
17
19
21
23
25
27
29
31
33
35


Hauteur d'eau (m)

Différence de hauteur (m)

RMS global : 0.34 m
RMS hautes eaux : 0.43 m
RMS basses eaux : 0.25 m

Comparison of altimetry and in situ water levels

PARAGUAY

DANUB

AMAZON

CONGO

Comparaison des hauteurs T/P 172 et des hauteurs à Brazzaville (d = 425 km)

287
288
289
290
291
292
293
294
295
296
297


Hauteur d'eau (m)

Différence de hauteur (m)

RMS global : 0.44 m
RMS hautes eaux : 0.46 m
RMS basses eaux : 0.42 m

CONGO

MEKONG RIVER

River ENV952MEKONGAS191 lon=105.070500 lat=16.096890

Water level (m)

2022 2003 2004 2005 2006 2007 2008 2009 2010

GDR Envisat
white: Envisat
yellow: Topex (2002-2005)
red: Jason, Topex 1993-2005
green: GFO

Lake Issykhul

GPS


Derived hydrological products
In addition to river level, discharge is also needed for various applications (water resource management, irrigation, flood/drought prediction, etc.)
Computation of discharge from altimetry-based water levels

River slope from altimetry + Manning’s equation = discharge $Q$

$$Q = \frac{1}{n} A S^{1/2} R^{2/3}$$

$R$ = hydraulic radius, $A$ = cross sectional area of flow, $P$ = wetted perimeter
$S$ = river slope, $n$ = roughness coefficient

Courtesy: C. Birkett
Global Reservoirs and Lakes Data Base

http://www.pecad.fas.usda.gov/cropexplorer/global_reservoirs

Delayed and near real time water level products

C. Birkett (GSFC/NASA)

PECAD data base (GSFC/NASA)

Lake Fitri, Chad

Lake Urmia, Iran

Lake Fitri Height Variations

TOPEX 9 Year Geo-referenced ROF Along Track Reference

Lake Urmia Height Variations

TOPEX 9 Year Geo-referenced ROF Along Track Reference
Rivers and Lakes ESA Data Base (P. Berry, UK) mainly ERS1/2 and ENVISAT

http://earth.esa.int/riverandlake

Lake Edouard (Africa)

Also: near real time water levels from ENVISAT in some regions
Soil moisture: SMOS

GRACE space gravimetry

Surface waters: altimetry

Total storage

Smith: soil moisture
Soil moisture from SMOS in western Africa during March 2010

Soil moisture values
2010/03/03 - 2010/03/23

Brightness temperatures
2010/03/03 - 2010/03/23

SMOS sees Pakistan floods (summer 2010)

17 July 2010

28 July 2010

4 August 2010

30 July 2010
Australian floods seen by SMOS

December 2010

Time provided in UTC Time

Surface waters: altimetry

Soil moisture: SMOS

Total storage

GRACE space gravimetry
GRACE space gravimetry (2002- )
→ time-variable gravity field
→ surface mass redistribution

Temporal resolution: ~ 1 month
Spatial resolution: 300-400 km

Permanent component
99% of the observed geoid;
Related to solid Earth’s structure

Temporal variations
• surface mass redistributions:
atmosphere, oceans, land waters, ice sheets
• Post-Glacial Rebound

GEOID HEIGHT

\[ \delta N(s,t) = G \iiint \frac{dm(r,t)}{|r-s|} \]

GRACE space gravimetry measures ice sheets mass change

Greenland ice mass (Gt)

Ice mass loss since 2002

-230 +/- 33 Gt/yr

+0.6 mm/yr

GRACE web site
East African lakes region

GRACE trends (2002-2006) in total water storage

blue = water deficit

Water storage over the Victoria lake

East African Lakes

(Altimetry and space gravimetry)

Lake Victoria

Lake level from altimetry


Drought 2005-2006

Water storage from GRACE

Becker et al. 2010
Amazon Basin

Trend in total water storage (2003-2008) (GRACE)

Blue = water deficit
Red = water excess

Surface water volume change from multi-satellite techniques:
Combining surface water extent and altimetry-derived water height

floodplains/inundation using multi-satellite technique
water level time series
Topex/Envisat

Estimation of surface water volume variations

Rio Negro basin
Altimeter track (T/P)
Altimeter station
In situ gauge station

Frappart et al., JGR, 2008;
Papa et al., 2006, 2008 GRL; Papa et al., 2010, Prigent et al., 2007 JGR;
Groundwater storage variations in various large aquifers using GRACE data

- GRACE mission (2002 to present)
- Estimation of the Total Water Storage (TWS)
- Surface and near-surface layers: Land Surface Models (GLDAS, ISBA, WGHM)
- Combination of GRACE and LSM to estimate GWS variations

Rio Negro (Amazonia)
Groundwater storage variations in various large aquifers using GRACE data

Water storage trend in the Ganges Basin
2002-2009 (cm/yr)

Rodell et al., 2009; Tiwari & Wahr, 2009
GRACE also sees ground water variations

Canning Aquifer Australia
(Blue = water deficit)

Ground water loss: ~10 km³/yr

Altimetry-based global mean sea level (1993-2011)

Rate of sea level rise (1993-2011)
3.3 ± 0.4 mm/yr
Nerem et al. (2010)

**MEI: Multivariate ENSO Index**

What processes control the observed correlation between interannual sea level and ENSO?:

- Ocean heat content?
- Land water storage?
Detrended global mean sea level
Land water storage from the ISBA-TRIP model

Sea level data from GSFC
(Beckley et al.)

Sea level data from CLS
(Ablain et al.)

Llovel et al., 2010

Contribution to sea level
of total water storage change
in the Amazon and La Plata basins
(and sum of both)

Expressed in equivalent sea level
(mm)
Comparison between north Pacific mass and sum of all hydrographic basins
120°E - american coasts and 0°N - 60°N

Sea level (mm)

Time (year)

Global Sea Level Drops 6 mm in 2010

Slope = 3.2 mm/year

El Nino '10 becomes La Nina '10

Mean Sea Level (cm)

Sea level data courtesy of S. Nerem, University of Colorado
GRACE-based water storage trend 2002-2009 (km$^3$/yr)

Net trend: + 80 +/- 27 km$^3$/yr

Sea level equivalent: -0.2 +/- 0.1 mm/yr

(Llovel et al., 2010)
SWOT: The Surface Water and Ocean Topography

Objectives

Hydrology:

- Measurement of water levels and storage of all types of surface water bodies (lakes, rivers, wetlands) of size >100mx100m
- Revisit time: 3 days and 22 days
- Launch: 2018-2019

CNES/NASA mission

SWOT coverage

- SWOT = Water mask + water elevation (and river slope) with 2 or more observations per 22 days

Nadir altimeter coverage
Lakes and SWOT

- Nadir altimeters miss more than 60% of lakes and can see area > 100 km² -> see only 15% of the global lake storage change
- SWOT = global coverage and see area > 250mx250m -> see 65% of the global lake storage change

Transboundary basins: Ganges

- Distance virtual station/gage = 530 km
- RMSE 5-day forecast/in-situ = 0.7 m
A GRACE Follow-On mission is crucially needed

Future developments

- Space observations (+ in situ)
  - Soil moisture
  - Water level
  - Discharge
  - Snow
  - Precipitation
  - Other data: DEM, Land use, Vegetation type...

- Water storage
- Data processing
  - Primary and derived hydrological products
  - Validation
- Modelling

- Water budget
- River basin scale

Data base of hydrological products

Dedicated future space missions

Hydrodynamical Functioning
Thanks for your attention
GRACE Level-2 Products

Torsten Mayer Guerr, Frank Flechtner

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

Content Part 2

- GRACE Measurement Principle: Level-1B Data
- Dynamic Approach to derive Level-2 GRACE Satellite-only Models („GSM“)
- AOD1B as a „special“ Background Model
- GRACE Level-2 „Gax“ Products
- RL05 Reprocessing at GFZ (with remarks on degree 2 coefficients)
- Summary
**GRACE Mission Concept**

- Observation of gravitationally caused orbit perturbations along the common line of sight (COM-COM) of a twin satellite pair by high-low (GPS) + low-low (K-Band) Satellite-to-Satellite Tracking (SST)
- Observation of non-gravitational accelerations by 3D Accelerometer (SuperSTAR)
- Observation of satellite and instrument orientation by Star Cameras
- Validation of GPS-derived orbit by Satellite Laser Ranging (SLR)

**GRACE Level-0 to Level-1B**

- Processing of Level-0 Raw Data (RDC 4/d) @ JPL
  - L0 Raw Data
  - Reversible „Reformatting“
    - Irreversible Transform.
      - GPS time scale
      - Outlier detection
      - Low pass filtering (10 Hz to 1/0.2Hz)
  - L1B-Data
- Input for Level-2 gravity field determination by GRACE SDS and other centers
Accelerometer Level-1B (ACC1B)

- servo-driven, capacitive accelerometer (ONERA)
- 3 axis measure linear and angular accelerations
- Specification: $\sigma_{X,\text{ACC}} = 10^{-9}$ [m/s$^2$/Hz], $\sigma_{Y,\text{ACC}} = \sigma_{Z,\text{ACC}} = 10^{-10}$ [m/s$^2$/Hz]
  (flight and radial direction)

"Good" correlation with classical surface force modelling (atmospheric drag, solar radiation, albedo) ...

.... linear accelerations from AOCS
Accelerometer Level-1B (ACC1B)

- Obviously lower accuracy than specified:
  - Radial, Along-Track ≈ 8 x Specification
  - Cross-Track ≈ 2 x Specification

- Problems/Causes:
  - intensive Attitude and Orbit Control (AOCs)
  - "Twangs" (vibrations of teflon foil at the bottom side of the satellites) due to albedo/magnetic field...
  - Unknown instrument errors
  - Combination of 2 star camera heads (accuracy to be rechecked with L1B V.02)
  - Others?

K-Band Level-1B (KBR1B)

- Dual one-way distance measurement based on phase measurement in K/Ka-Band
- $\sigma_{Range} = 10 [\mu m]$, $\sigma_{Range-Rate} = 1 [\mu m/s]$
K-Band Level-1B (KBR1B)

- Accuracy as expected or even better:
  - K-Band Range (KRA) $= 1 \mu$m
    ($<<$ specification $10 \mu$m )
  - K-Band Range-Rate (KRR) $\approx 0.2 \mu$m/s
    ($<<$ specification $= 1 \mu$m/s)

Level-1 Instrument Data: Further Reading

- Algorithm Theoretical Basis Document for GRACE Level-1B Data Processing V1.2
  - Sien-Chong Wu
  - Gerhard Knutingsa
  - Willy Bertiger
  - May 8, 2006

- GRACE Level 1B Data Product User Handbook
  - Update with corrections for treatment of KBR1B
  - Signal to Noise Ratio
  - Formats
  - Interpretation of Data
  - etc.
Dynamical Method: General

- “Classical” method for adjustment of orbits and/or gravity models from satellite data
- Combination of
  - Numerical integration (equation of motion → orbit) plus
  - Methods of parameter adjustment („Least squares method‟)
- Advantages
  - High flexibility (observation types, models...) and accuracy
  - Adjustment of observation series with gaps
  - Simultaneous adjustment of heterogeneous satellite data (integrated analysis)
- Disadvantages
  - High numerical effort (especially for gravity field determination)
  - No analytical description of functional behaviors

Dynamical Method: Functional Relationships

Functional relationship between (GPS/KBR) SST observations \( \mathbf{b} \) and looked for gravity field model coefficients and other parameters \( \mathbf{p} \) is not linear:

\[
\mathbf{b} = f(t, \mathbf{r}, \dot{\mathbf{r}}, \mathbf{p})
\]

Orbital parameters
Gravity coefficients
Instrument parameters

Linearized Gauß-Markoff-Model:

\[
\mathbf{l} = \mathbf{A} \Delta \mathbf{p} + \mathbf{v}
\]

Correction of observations
looked for adjustment values: \( \Delta \mathbf{p} = \mathbf{p} - \mathbf{p}_0 \)

partial derivatives

Vectors of discrepancies resp. reduced observations „o-c” \( \mathbf{l} = \mathbf{b} - \mathbf{b}_0 \)

\( \mathbf{p}_0 \): Approximation of parameters (e.g. from models)

\( \mathbf{b}_0 \): Theoretical observations
**Dynamical Method: Functional Relationships**

Functional relationship between (GPS/KBR) SST observations \( b \) and looked for gravity field model coefficients and other parameters \( p \) is not linear:

\[
b = f(t, r, \dot{r}, p)
\]

Orbital parameters
Gravity field coefficients
Instrument parameters

**Adjustment of \( \Delta p \) by minimizing \( v^TPv \):**

\[
\Delta p = (A^T P A)^{-1} A^T P \Delta l
\]

\[
P = D(b)^{-1} = \sigma^2 Q_{bb}
\]

\( p_0 \): Approximation of parameters (e.g. from models)

\( b_0 \): Theoretical observations

**Dynamical Method: Approximation of Observations**

Get approximated/theoretical observations \( b_0 \) by numerical integration of the equation of motion (initial value problem):

\[
\dot{r} = -\frac{GM}{r^3} r + a_g + a_{ng} + a_e
\]

Central/KeplerTerm
Relativistic/empirical accelerations
Non-gravitational accelerations
Additional gravitational accelerations

\( \Rightarrow \) dynamical orbit \( r_0(t), \dot{r}_0(t) \)

\( \Rightarrow \) theoretical observations \( b_0 = f(t, r_0, \dot{r}_0, p_0) \)
Dynamical Method: (non)Gravitational Accelerations

\[ \ddot{\mathbf{r}} = -\frac{GM}{r^3} \mathbf{r} + \mathbf{a}_g + \mathbf{a}_{ng} + \mathbf{a}_e \]

Non gravitational accelerations from accelerometer observations:

\[ \mathbf{a}_{ng} \approx \mathbf{b}_{\text{AddKonst}} + \mathbf{M}_{3 \times 3} \cdot \mathbf{a}_{\text{SuperSTAR}} \]

Needs adjustment of instrument parameters (biases and scales)

Gravitational accelerations as sum of gradients of \( n \) potential functions:

\[ \mathbf{a}_g = \sum_{i=1}^{n} \nabla \mathbf{V}_i (r, \theta, \lambda, t, p_0) \]

Gravity Potentials for GFZ EIGEN Solutions

<table>
<thead>
<tr>
<th>Potential</th>
<th>Parameter ( p_0 )</th>
<th>Potential</th>
<th>Parameter ( p_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static Gravity Field</td>
<td>( C_{nm}, S_{nm} ) Unknows ( \Delta C_{nm}(t), \Delta S_{nm}(t) )</td>
<td>Third Bodies</td>
<td>Tabled Coordinates of Planets</td>
</tr>
<tr>
<td>Earth Tides</td>
<td>Amplitudes + Phases ( \hat{C}<em>{smn}^{\pm}, \hat{S}</em>{smn}^{\pm} )</td>
<td>Ocean Tides</td>
<td>Amplitudes + Phases ( \hat{D}<em>{smn}^{\pm}, \hat{D}</em>{smn}^{\pm} )</td>
</tr>
<tr>
<td>Short-term Mass Variations in Atmosphere and Oceans</td>
<td>6h Correction Terms ( \Delta C_{nm}(t), \Delta S_{nm}(t) )</td>
<td>Atmospheric Tides</td>
<td>Amplitudes + Phases ( \hat{A}<em>{smn}^{\pm}, \hat{S}</em>{smn}^{\pm} )</td>
</tr>
<tr>
<td>Pole Tides (Solid Earth, Ocean)</td>
<td>Pole Coordinates ( x_p(t), y_p(t) \rightarrow \Delta C_{j\ell}(t), \Delta S_{j\ell}(t) )</td>
<td>Periodic and Secular Variations e.g. in Hydrosphere or Cryosphere</td>
<td>Models, e.g. ( \hat{C}<em>{nm}, \hat{S}</em>{nm} )</td>
</tr>
</tbody>
</table>
Summary: Level-2 Processing

Static Gravity Model

Reduction, Accumulation and Solution

Monthly Model 1
Monthly Model 2
Monthly Model N

L2-Products >> 30/31 d = 30/31 d

Background Models, Data and Standards:
- Static Gravity Field
- AOD1B
- GPS Constellations

Normal Equation Systems
A^T PA, A^T PI

Dynamic Orbit Determination
- Initial Values for Linearization
- Data Screening

Convergence ?

Nominal Arc Length: 1 Day

L1B-Data
GPS-SST, KRR-SST, ACC, SCA

Sources of Gravity Variations

High Frequency Variations (Models):
- Ocean Tides: FES2004, EOT11a...
- Atmospheric Tides: S1/S2 (Bode & Biancale, ...)
- Non-tidal Atmosphere: see next
- Non-tidal Ocean: see next
- Continental Water Cycle: WGHM, ... or GRACE (from multiple years or daily output)

Seasonal Variations (GRACE results):
- Continental Water Cycle
- Ice Mass Loss (Greenland/Antarctica)
- Surface and Deep Ocean Currents, Ocean Mass

Gravity variations in terms of geoid height variations estimated from surface pressure, ocean bottom pressure and continental water heights.
AOD1B: Concept Atmosphere

ECMWF, NCEP, DWD (6h, 0.5° analysis results)

- Compute Block Mean

- Surface Data Pressure & Geopotential Heights
- Multi Level Data Temperature & Humidity

- Subtract

- Residual Surface Pressure

- Compute Gravity Coefficients

- Long Term Mean Fields (for each SHS Degree)

- Integrate over Atmospheric Column

- Subtract

- Residual Fields (per SHS Degree)

- Compute Gravity Coefficients

Gravity Coefficients (6 hourly series)

AOD1B: Oceanic Part (IB vs non-IB)

- Inverse barometric effect (IB) assumes that atmospheric pressure is fully compensated by the ocean: Atmospheric ("SP/VI") + oceanic mass variation = 0.

- Assumption is not sufficient for modern satellite missions: Need ocean models!

Only "Quasi-Compensation"
AOD1B: Concept Ocean

ECMWF, NCEP, DWD
(6 hourly analysis results)

- Sea level pressure
- 10m u Velocity
- 10m v Velocity

Note: The atmosphere is added in a next step to Sea Level Pressure

Sea Level Pressure

Run barotropic ocean model

Sea Level Pressure

Subtract

Residual Sea Level Pressure

Run baroclinic ocean model

Sea Level Pressure

Subtract

Residual Sea Level Pressure

Compute Gravity Coefficients

Gravity Coefficients (6 hourly series)

Min, Max, Mean and wRMS of 6h RL04 for 2007-2008

"ATM"+"OCN"="GLO"

January 2008

"OBA"

Land: 0
Ocean: SP+OMCT

ECMWF, NCEP, DWD
(6 hourly analysis results)

- Sea Level Pressure
- 2m Air Temperature
- 10m u Velocity
- 10m v Velocity
- Freshwater flux (P-E) from ECMWF forecast

ECMWF, NCEP, DWD
(6 hourly analysis results)

- Sea Level Pressure
- 2m Air Temperature
- 10m u Velocity
- 10m v Velocity
- Freshwater flux (P-E) from ECMWF forecast
Mean GAx Products

For every (e.g. weekly/monthly) gravity field model a corresponding mean (GA = GRACE Average) for all 4 AOD1B components is provided.

Only days used within gravity field determination period are averaged.

GAA: Mean of the 6h "atm":  
   Land: VI-Vimean  
   Ocean: VI-Vimean

GAB: Mean of the 6h "ocn":  
   Land: 0  
   Ocean: OMCT-OMCTmean

(this two products are more or less only informal)

GAC: Mean of the 6h "glo":  
   Land: VI-Vimean  
   Ocean: (VI-Vimean)+(OMCT-OMCTmean)

(this is the mean product of 6h variations which have been subtracted during gravity field determination)

GAD: Mean of the 6h "oba":  
   Land: 0  
   Ocean: (SP-SPmean)+(OMCT-OMCTmean)

(Since RL04 as should be closer to ocean bottom pressure and to decrease leakage from land)

Difference using/not using AOD1B/GAC in Gravity Field Determination

Without/with filter

Neglecting AOD1B has large impact!  
- Stripes  
- wRMS = 0.5 mm
AOD1B (RL05): higher temporal and spatial resolution of OMCT

RL04 – RL05 (interpolated @ 1°)

- RL05 shows enhanced gradients in most western boundary current regions (Gulf Stream, Kuroshio current)
- Subtle differences in equatorial current systems (were rather overestimated in RL04) and in the Arctic
- Strong differences in ACC region: meridional gradients are much stronger in RL05, topography apparent in the flow path

AOD1B Validation: **Daily** AOD RL04/RL05 Correlations with OBP (2008)

<table>
<thead>
<tr>
<th>Corr. 2008</th>
<th>RL04</th>
<th>RL05</th>
<th>Improvement [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Daily GAC</td>
<td>Daily GAD</td>
<td>Daily GAC</td>
</tr>
<tr>
<td>All</td>
<td>0.42</td>
<td>0.44</td>
<td>0.59</td>
</tr>
<tr>
<td>ACC</td>
<td>0.46</td>
<td>0.45</td>
<td>0.54</td>
</tr>
<tr>
<td>DART</td>
<td>0.39</td>
<td>0.40</td>
<td>0.57</td>
</tr>
<tr>
<td>FRAM</td>
<td>0.56</td>
<td>0.61</td>
<td>0.72</td>
</tr>
</tbody>
</table>

... shows large model improvements
AOD1B Validation: **Daily AOD RL04/05/ITG Correlations with OBP (2008)**

<table>
<thead>
<tr>
<th>Corr. 2008</th>
<th>RL04</th>
<th>RL05</th>
<th>Improvement [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>GAC</td>
<td>GAD</td>
<td>GAC</td>
<td>GAD</td>
</tr>
<tr>
<td>All</td>
<td>0.42</td>
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<td>0.59</td>
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<td>0.72</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Corr. 2008</th>
<th>RL04</th>
<th>ITG2010</th>
<th>Improvement [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>GAC</td>
<td>GAD</td>
<td>GSM+GAC</td>
<td>GSM+GAD</td>
</tr>
<tr>
<td>All</td>
<td>0.42</td>
<td>0.44</td>
<td>0.46</td>
</tr>
<tr>
<td>ACC</td>
<td>0.46</td>
<td>0.45</td>
<td>0.75</td>
</tr>
<tr>
<td>DART</td>
<td>0.39</td>
<td>0.40</td>
<td>0.37</td>
</tr>
<tr>
<td>FRAM</td>
<td>0.56</td>
<td>0.61</td>
<td>0.75</td>
</tr>
</tbody>
</table>

- ITG (GAC plus GRACE corrections) gives high correlations at sites with large OBP signal (ACC)
- AOD RL05 gives higher correlations at sites with small OBP signal (DART)
- Globally, AOD1B RL05 gives higher correlations (0.59 vs. 0.46)

---

**Monthly Comparisons to Ocean Bottom Pressure for RL04 (2003-2008)**

<table>
<thead>
<tr>
<th>DDK1 (530km) RL04</th>
<th>Stations with (OBP / GSM+GAC) SNR &gt; 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>GSM</td>
<td>GAC (RL04)</td>
</tr>
<tr>
<td>All (54)</td>
<td>0.02</td>
</tr>
<tr>
<td>ACC (6)</td>
<td>0.29</td>
</tr>
<tr>
<td>DART (7)</td>
<td>0.21</td>
</tr>
<tr>
<td>FRAM (6)</td>
<td>0.72</td>
</tr>
<tr>
<td>KESS (32)</td>
<td>-0.25</td>
</tr>
<tr>
<td>POL_ACCLAIM (3)</td>
<td>0.30</td>
</tr>
</tbody>
</table>

- Geocenter motion considered for GSM+GAC product (IIGOG time series)
- Only comparisons for long time series give a clear picture (RL05 available only for 2008)

- Model (GAC/D) only gives small correlations (room for improvement, see RL05)
- GAD correlations smaller than GAD (against theory, but improved in RL05 (see before))
- GRACE contributes to OBP (see daily correlations ITG2010): This is the reason why the user gets GSM and Gax!
AOD1B: Further Reading

All Details described in this Document ->

Also recommended:

Lecture Gummersbach 2007:
“De-aliasing von atmosphärischen und ozeanischen Kurzzeit-Massenvariationen für CHAMP, GRACE und GOCE: Stand und offene Fragen”

AOD1B quality control and status page:
http://www-app2.gfz-potsdam.de/pb1/op/grace/results/grav/g007_aod1b_r104.html

Level-2 Products: Further Reading

GRACE 327-734
(CSR-GR-03-01)
Gravity Recovery and Climate Experiment
Level-2 Gravity Field Product User Handbook
(Rev 2.0, February 26, 2007)

Provides information on GSM and Gax Level-2 data

GRACE 327-732
(GR-GFZ-FD-001)
Gravity Recovery and Climate Experiment
GRACE Gravity Field Solution Data Formats
(Rev. 1.1, November 27, 2003)

Format of Level-2 Products

GRACE 327-743
(GR-GFZ-01D-001)
Gravity Recovery and Climate Experiment
GRACE Level-2 Processing Standards Document
For Level-2 Product Release 01D
(Rev. 1.0, February 19, 2007)

For each center and each release: Applied background models & standards
RL05 Reprocessing @ GFZ: Processing Standards

- Improved L1B data & preprocessing:
  - L1B RL02 data
  - reduction of data elimination during preprocessing

- Improved GPS processing:
  - absolute antenna phase center variations, phase windup correction, GPS attitude model, IGS2008
  - GFZ derived GPS antenna maps for GRACE

- More dense accelerometer parameterization:
  - 1-hourly biases in radial, along-track and cross-track direction
  - no more scale factors

- Relative weighting of GPS-SST and KRR observations:
  - increased KRR weight (~ by a factor of 2)
  - gravity field parameters with n > 80 estimated only with KRR

RL05 Reprocessing @ GFZ: Improved Background Models

<table>
<thead>
<tr>
<th></th>
<th>Currently: RL04</th>
<th>New: RL05</th>
</tr>
</thead>
<tbody>
<tr>
<td>A priori Static Gravity Field</td>
<td>EIGEN-GL04C</td>
<td>EIGEN-6C</td>
</tr>
<tr>
<td>Time-variable Gravity Field</td>
<td>none</td>
<td>Trend/Annual/Semiannual Model derived from EIGEN-6C</td>
</tr>
<tr>
<td>Secular Rates</td>
<td>C_{20}, C_{30}, C_{40}, C_{21}, S_{21}</td>
<td>none</td>
</tr>
<tr>
<td>Ocean Tides</td>
<td>FES2004</td>
<td>EOT11a</td>
</tr>
<tr>
<td>Atmospheric Tides S1, S2</td>
<td>Bode-Biancale 2003</td>
<td>Bode-Biancale 2003</td>
</tr>
<tr>
<td>Atmospheric and Oceanic Non-tidal Mass Variations</td>
<td>AOD1B RL04</td>
<td>AOD1B RL05</td>
</tr>
<tr>
<td>Solid Earth &amp; Pole Tides</td>
<td>IERS2003</td>
<td>IERS2010</td>
</tr>
<tr>
<td>3rd Body Ephemerides</td>
<td>JPL DE403</td>
<td>JPL DE421</td>
</tr>
</tbody>
</table>
RL05 Reprocessing @ GFZ: Prefit Residuals 2008

GPS phase (44%)  
K-band range rate (33%)

RL05 Reprocessing @ GFZ: Error Characteristics 2008

(RL03: 17.5 x baseline)  
RL04: 15 x baseline  
RL05: 8 x baseline
RL05 Reprocessing @ GFZ: Error Characteristics 2008

**d/o 60x60: ~40-60%**

wRMS in terms of EWH [cm] w.r.t. ITG-Grace2010s

**RMS variability w.r.t. mean (EWH [cm], DDK3)**

![Graph showing RMS variability over time](image)

- Problem exists in all center solutions! Remains (pronounced) in GFZ RL05
- Not optimal solution for RL04: Substitute GRACE C20 coefficient by SLR derived value (see TN05)
- Plan for final RL05: Find reason or combine with SLR (GRGS approach)

---

RL05 Reprocessing @ GFZ: Unresolved 161d signal in C20

**365d**

182d

161d

![Power Spectral Density](image)

**Monthly Power Spectral Density**

- Monthly C20 GFZ RL04.plot
  - GFZ/GRACE RL04
  - GFZ/GRACE RL05
  - GRACE C20
  - C05 SLR RL04

**Month of Year 2008**

![Graph showing power spectral density over time](image)
• Already much better correlations for all centers
• Notable improvement with GFZ RL05

Summary Part 2

• During GRACE Level-2 gravity field determination all “known” gravity variations (trend, sannual, semi-annual, ...) are taken into account (models). Note: GIA (Global Isostatic Adjustment) model not yet used.
• Before provision to the users some (monthly mean) models are restored (e.g. hydrology, ice mass loss): impacts results mostly over land
• Over the oceans GRACE plus GAx products have to be used when comparing with in-situ OBP data! Here, also degree 1 coefficients (not provided by GRACE) have to be taken into account.
• GRACE C20 coefficient still shows spurious (unexplained) 161d signal
• A new (much improved) RL05 time series incl. a new AOD1B RL05 and reprocessed Level-1B instrument data covering the complete GRACE mission lifetime will be made available by the GRACE Science Data System on March 17, 2012 (10th anniversary of GRACE)
Spherical Harmonics & Gravity field

Torsten Mayer-Gürr

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

Content

Part 1:
- Spherical Harmonics
- Gravity field
  - Geoid heights
  - Gravity anomalies
  - Gravity disturbances
  - Total water storage
- Degree Variances
- Upward Continuation

Part 2: (Frank Flechtner)
- GRACE Processing & GRACE Products
Approximation of functions on the sphere

Approximation

Approximation with a polynomial of degree $n$:

$$ f(x) = a_0 p_0(x) + a_1 p_1(x) + \cdots + a_n p_n(x) $$

$$ p_n(x) = x^n $$

Approximation of a periodic function with a Fourier series:

$$ f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos \left( \frac{2\pi}{T} t \right) + s_n \sin \left( \frac{2\pi}{T} t \right) $$

Approximation of a function on the sphere with spherical harmonics:

$$ f(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) $$
Approximation with Spherical harmonics

Spherical harmonics:

\[ Y(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

Approximation on the sphere:

\[ \|f(\lambda, \vartheta) - Y(\lambda, \vartheta)\|^2 = \int_{\Phi} (f(\lambda, \vartheta) - Y(\lambda, \vartheta))^2 d\Phi = \min \]

Norm and scalar product:

\[ \|f\|^2 = \int_{\Phi} f^2(\lambda, \vartheta) d\Phi \quad \langle f, g \rangle = \int_{\Phi} f(\lambda, \vartheta) \cdot g(\lambda, \vartheta) d\Phi \]

With orthogonal basis functions:

\[ \langle Y_{nm}, Y_{n'm'} \rangle = 0 \quad \text{für } n \neq n' \text{ oder } m \neq m' \]

The solution is simple:

\[ a_{nm} = \frac{1}{\|Y_{nm}\|} \langle f, Y_{nm} \rangle \]

Approximation with Spherical harmonics

Expansion into a series of spherical harmonics:

\[ f(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

with the coefficients:

\[ a_{nm} = \frac{1}{\|Y_{nm}\|} \int_{\Phi} f(\lambda, \vartheta) \cdot Y_{nm}(\lambda, \vartheta) d\Phi \]

Norm of the basis functions:

\[ \|Y_{nm}\|^2 = \int_{\Phi} Y_{nm}^2(\lambda, \vartheta) d\Phi \]

The norm is arbitrary:

Not normalized (e.g. mathematics):

\[ \|Y_{nm}(\lambda, \vartheta)\|^2 = \frac{4\pi}{2n+1} (2-\delta_{nm}) \frac{(n+|m|)!}{(n-|m|)!} \]

Schmidt semi-normalized (e.g. magnetics):

\[ \|Y_{nm}(\lambda, \vartheta)\|^2 = \frac{4\pi}{2n+1} \]

Fully normalized (e.g. gravity):

\[ \|Y_{nm}(\lambda, \vartheta)\|^2 = 4\pi \]
Approximation of a function on the sphere

\[ f(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

**Arrangement of coefficients in a triangle**

- **Degree \( n = 0 \)**
  - \( a_{00} \)
- **Degree \( n = 1 \)**
  - \( a_{1,-1}, a_{10}, a_{11} \)
- **Degree \( n = 2 \)**
  - \( a_{2,-2}, a_{2,-1}, a_{20}, a_{21}, a_{22} \)
- **Degree \( n = 3 \)**
  - \( a_{3,-3}, a_{3,-2}, a_{3,-1}, a_{30}, a_{31}, a_{32}, a_{33} \)
- **Degree \( n = 4 \)**
  - \( a_{4,-4}, a_{4,-3}, a_{4,-2}, a_{4,-1}, a_{40}, a_{41}, a_{42}, a_{43}, a_{44} \)
- **...**
  - \( a_{5,-5}, a_{5,-4}, a_{5,-3}, a_{5,-2}, a_{5,-1}, a_{50}, a_{51}, a_{52}, a_{53}, a_{54}, a_{55} \)

**Approximation of functions on the sphere**
Approx. with spherical harmonics

\[ f(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

<table>
<thead>
<tr>
<th>Degree ( n )</th>
<th>Number of coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>25</td>
</tr>
<tr>
<td>8</td>
<td>81</td>
</tr>
<tr>
<td>16</td>
<td>289</td>
</tr>
<tr>
<td>30</td>
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<tr>
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<td>58081</td>
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Approx. with spherical harmonics

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Approx. with spherical harmonics

\[ f(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \theta) \]

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Approx. with spherical harmonics

\[ f(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

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<tr>
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</tr>
</tbody>
</table>
**Basis functions**

\[
Y_{n,m} (\lambda, \vartheta) = \cos(m\lambda) \, P_{nm} (\cos \vartheta)
\]

\[
Y_{n,-m} (\lambda, \vartheta) = \sin(m\lambda) \, P_{nm} (\cos \vartheta)
\]

**Approximation of a function on the sphere**

\[
f(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm} (\lambda, \vartheta)
\]

**Approximation of a function on the sphere**

\[
f(\lambda, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda) \right) P_{nm} (\cos \vartheta)
\]

**Computation of the basis functions**

\[
Y_{n,m} (\lambda, \vartheta) = P_m^n (\cos \vartheta) \cos m\lambda
\]

\[
Y_{n,m} (\lambda, \vartheta) = P_m^n (\cos \vartheta) \sin m\lambda
\]

**Legendre functions**

\[
P_0^0 (t) \quad P_1^0 (t) \quad P_1^1 (t)
\]

\[
P_2^0 (t) \quad P_2^1 (t) \quad P_2^2 (t)
\]

\[
P_3^0 (t) \quad P_3^1 (t) \quad P_3^2 (t) \quad P_3^3 (t)
\]

\[
P_4^0 (t) \quad P_4^1 (t) \quad P_4^2 (t) \quad P_4^3 (t) \quad P_4^4 (t)
\]

**Recursion formula**

\[
P_0^0 (t) = 1
\]

\[
P_n^0 (t) = a_n \cdot \sqrt{1-t^2} \cdot P_{n-1}^{n-1} (t)
\]

\[
P_{n-1} (t) = b_n^m \cdot t \cdot P_{n-1}^m (t)
\]

\[
P_m (t) = b_n^m \cdot t \cdot P_{n-1}^m (t) - c_n^m \cdot P_{n-2}^m (t)
\]

**Factors (normalized)**

\[
a_0 = \frac{\sqrt{3}}{2n} \quad a_n = \frac{\sqrt{3}}{2n+1}
\]

\[
b_n^m = \frac{(2n+1)(2n-1)}{(n+m)(n-m)}
\]

\[
c_n^m = \frac{(2n+1)(n-m-1)(n+m-1)}{(2n-3)(n+m)(n-m)}
\]
### Basis functions

#### Basis functions degree $n = 4$:

<table>
<thead>
<tr>
<th>Degree</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$C_{00}(\lambda, \vartheta) = P_4^0(\cos \vartheta)$</td>
</tr>
<tr>
<td>1</td>
<td>$C_{10}(\lambda, \vartheta) = \cos(\lambda) \cdot P_4^1(\cos \vartheta)$</td>
</tr>
<tr>
<td>2</td>
<td>$S_{10}(\lambda, \vartheta) = \sin(\lambda) \cdot P_4^1(\cos \vartheta)$</td>
</tr>
<tr>
<td>2</td>
<td>$C_{20}(\lambda, \vartheta) = \cos(2\lambda) \cdot P_4^2(\cos \vartheta)$</td>
</tr>
<tr>
<td>3</td>
<td>$S_{20}(\lambda, \vartheta) = \sin(2\lambda) \cdot P_4^2(\cos \vartheta)$</td>
</tr>
<tr>
<td>3</td>
<td>$C_{30}(\lambda, \vartheta) = \cos(3\lambda) \cdot P_4^3(\cos \vartheta)$</td>
</tr>
<tr>
<td>4</td>
<td>$S_{30}(\lambda, \vartheta) = \sin(3\lambda) \cdot P_4^3(\cos \vartheta)$</td>
</tr>
<tr>
<td>4</td>
<td>$C_{40}(\lambda, \vartheta) = \cos(4\lambda) \cdot P_4^4(\cos \vartheta)$</td>
</tr>
<tr>
<td>4</td>
<td>$S_{40}(\lambda, \vartheta) = \sin(4\lambda) \cdot P_4^4(\cos \vartheta)$</td>
</tr>
</tbody>
</table>
### Basis functions

**Basis functions degree** $n = 4$:

- $C_{40}(\lambda, \vartheta) = P_4^0(\cos \vartheta)$
- $C_{41}(\lambda, \vartheta) = \cos(1\lambda) \cdot P_4^1(\cos \vartheta)$
- $S_{41}(\lambda, \vartheta) = \sin(1\lambda) \cdot P_4^1(\cos \vartheta)$
- $C_{42}(\lambda, \vartheta) = \cos(2\lambda) \cdot P_4^2(\cos \vartheta)$
- $S_{42}(\lambda, \vartheta) = \sin(2\lambda) \cdot P_4^2(\cos \vartheta)$
- $C_{43}(\lambda, \vartheta) = \cos(3\lambda) \cdot P_4^3(\cos \vartheta)$
- $S_{43}(\lambda, \vartheta) = \sin(3\lambda) \cdot P_4^3(\cos \vartheta)$
- $C_{44}(\lambda, \vartheta) = \cos(4\lambda) \cdot P_4^4(\cos \vartheta)$
- $S_{44}(\lambda, \vartheta) = \sin(4\lambda) \cdot P_4^4(\cos \vartheta)$
### Basis functions degree $n = 4$:

- $C_{40}(\lambda, \vartheta) = P^0_4(\cos \vartheta)$
- $C_{41}(\lambda, \vartheta) = \cos(1\lambda) \cdot P^1_4(\cos \vartheta)$
- $S_{41}(\lambda, \vartheta) = \sin(1\lambda) \cdot P^1_4(\cos \vartheta)$
- $C_{42}(\lambda, \vartheta) = \cos(2\lambda) \cdot P^2_4(\cos \vartheta)$
- $S_{42}(\lambda, \vartheta) = \sin(2\lambda) \cdot P^2_4(\cos \vartheta)$
- $C_{43}(\lambda, \vartheta) = \cos(3\lambda) \cdot P^3_4(\cos \vartheta)$
- $S_{43}(\lambda, \vartheta) = \sin(3\lambda) \cdot P^3_4(\cos \vartheta)$
- $C_{44}(\lambda, \vartheta) = \cos(4\lambda) \cdot P^4_4(\cos \vartheta)$
- $S_{44}(\lambda, \vartheta) = \sin(4\lambda) \cdot P^4_4(\cos \vartheta)$
Basis functions

Basis functions degree $n = 4$:

\[ C_{40}(\lambda, \vartheta) = P_4^0(\cos \vartheta) \]
\[ C_{41}(\lambda, \vartheta) = \cos(1\lambda) \cdot P_4^1(\cos \vartheta) \]
\[ S_{41}(\lambda, \vartheta) = \sin(1\lambda) \cdot P_4^1(\cos \vartheta) \]
\[ C_{42}(\lambda, \vartheta) = \cos(2\lambda) \cdot P_4^2(\cos \vartheta) \]
\[ S_{42}(\lambda, \vartheta) = \sin(2\lambda) \cdot P_4^2(\cos \vartheta) \]
\[ C_{43}(\lambda, \vartheta) = \cos(3\lambda) \cdot P_4^3(\cos \vartheta) \]
\[ S_{43}(\lambda, \vartheta) = \sin(3\lambda) \cdot P_4^3(\cos \vartheta) \]
\[ C_{44}(\lambda, \vartheta) = \cos(4\lambda) \cdot P_4^4(\cos \vartheta) \]
\[ S_{44}(\lambda, \vartheta) = \sin(4\lambda) \cdot P_4^4(\cos \vartheta) \]
### Basis functions degree $n = 4$:

<table>
<thead>
<tr>
<th>$C_{40}(\lambda, \vartheta)$</th>
<th>$P_{4}^{0}(\cos \vartheta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{41}(\lambda, \vartheta)$</td>
<td>$\cos(1\lambda) \cdot P_{4}^{1}(\cos \vartheta)$</td>
</tr>
<tr>
<td>$S_{41}(\lambda, \vartheta)$</td>
<td>$\sin(1\lambda) \cdot P_{4}^{1}(\cos \vartheta)$</td>
</tr>
<tr>
<td>$C_{42}(\lambda, \vartheta)$</td>
<td>$\cos(2\lambda) \cdot P_{4}^{2}(\cos \vartheta)$</td>
</tr>
<tr>
<td>$S_{42}(\lambda, \vartheta)$</td>
<td>$\sin(2\lambda) \cdot P_{4}^{2}(\cos \vartheta)$</td>
</tr>
<tr>
<td>$C_{43}(\lambda, \vartheta)$</td>
<td>$\cos(3\lambda) \cdot P_{4}^{3}(\cos \vartheta)$</td>
</tr>
<tr>
<td>$S_{43}(\lambda, \vartheta)$</td>
<td>$\sin(3\lambda) \cdot P_{4}^{3}(\cos \vartheta)$</td>
</tr>
<tr>
<td>$C_{44}(\lambda, \vartheta)$</td>
<td>$\cos(4\lambda) \cdot P_{4}^{4}(\cos \vartheta)$</td>
</tr>
<tr>
<td>$S_{44}(\lambda, \vartheta)$</td>
<td>$\sin(4\lambda) \cdot P_{4}^{4}(\cos \vartheta)$</td>
</tr>
</tbody>
</table>

### Basis functions degree $n = 20$:

| $C_{20,5}(\lambda, \vartheta)$ | $\cos(5\lambda) \cdot P_{20}^{5}(\cos \vartheta)$ |

---

Basis functions
Basis functions degree $n = 40$:

\[
C_{40,20} (\lambda, \vartheta) = \cos(20 \lambda) \cdot P_{40}^{20} (\cos \vartheta)
\]

Basis functions

- **Degree $n = 0$**
- **Degree $n = 1$**
- **Degree $n = 2$**
- **Degree $n = 3$**
- **Degree $n = 4$**
- **Degree $n = 5$**

\[
Y_{n,m} (\lambda, \vartheta) = P_{n}^{m} (\cos \vartheta) \cos m \lambda \\
Y_{n,-m} (\lambda, \vartheta) = P_{n}^{m} (\cos \vartheta) \sin m \lambda
\]
Coefficient triangle

Accuracies ITG-Grace2010s (2008-03)

Gravity field
Gravity field:
\[ \mathbf{g}(\mathbf{r}) = \begin{pmatrix} g_x(\mathbf{r}) \\ g_y(\mathbf{r}) \\ g_z(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \frac{m}{r^3} \end{pmatrix} \]

Conservative vector field \( \Leftrightarrow \) Potential function exists:
\[ \mathbf{g}(\mathbf{r}) = \nabla V(\mathbf{r}) = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix} \quad V(\mathbf{r}) = \begin{pmatrix} \frac{m^2}{r^3} \end{pmatrix} \]

Source free in outer space
\[ \text{div} \mathbf{g} = \nabla \cdot \mathbf{g} = 0 \]
Laplace equation (harmonic func.)
\[ \nabla \cdot \nabla V = \Delta V = 0 \]
\[ \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \]

Harmonics continuation

Potential in outer space
\[ V(\lambda, \vartheta, r) = \frac{R(R^2 - r^2)}{4\pi} \int \frac{1}{l^3} V(\lambda, \vartheta) d\Omega \]
with \( l = \sqrt{R^2 + r^2 - 2Rr \cos \psi} \)

Harmonic continuation
\[ \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \]

Potential at Earth’s surface
\[ V(\lambda, \vartheta)|_{\Omega} \]
Harmonics continuation

Potential in outer space (solid spherical harmonics)

\[ V(\lambda, \vartheta, r) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^{n+1} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

Harmonic continuation

\[ \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \]

Potential at Earth’s surface (in spherical harmonics)

\[ V(\lambda, \vartheta) = \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

Gravity

Gravity acceleration at Earth’s surface

\[ g = 9.78 \frac{m}{s^2} \ldots 9.83 \frac{m}{s^2} \]

\[ \pm 0.0004 \frac{m}{s^2} \]
Gravity disturbances

ITG-Grace2010s (difference 2008-09, 2008-03)

Gravity disturbances

ITG-Grace2010s (difference 2008-09, 2008-03)
Gravity disturbances

ITG-Grace2010s (difference 2008-09, 2008-03)

ITG-Grace2010s (difference 2008-09, 2008-03), Gaussian filter 300 km
Gravity disturbances

ITG-Grace2010s (difference 2008-09, 2008-03), Gaussian filter 400 km

[Map showing gravity disturbances]

Gravity disturbances

ITG-Grace2010s (difference 2008-09, 2008-03), Gaussian filter 500 km

[Map showing gravity disturbances]
### Gravity field

**Sphere**
\[
g = \frac{GM}{R^2} \approx 10 \frac{m}{s^2} \quad C_{00} = 1
\]

**Flattening**
\[
g = \pm 0.03 \frac{m}{s^2} \quad C_{20} = -0.0004842
\]

**Static part**
\[
\delta g \approx 0.001 \frac{m}{s^2} = 100 \text{mGal}
\]

**Time variable part**
\[
\Delta \delta g \approx 0.000001 \frac{m}{s^2} = 0.01 \text{mGal}
\]

### Gravity field functionals

**Disturbing potential**
\[
T = V - U
\]

**Gravity**
\[
g = \|\nabla V\|
\]

**Geoid heights**
\[
V(\mathbf{r}) \approx V(\mathbf{r}_0) + \|\nabla V\| d\mathbf{r}
\]
\[
N = d\mathbf{r} = \frac{T}{\|\nabla V\|} = \frac{T}{g}
\]

**Geoid heights changes**
\[
\Delta N = \frac{\Delta T}{g}
\]

**Normal potential**
\[
T(\lambda, \theta, r) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^{n+1} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \theta)
\]

**Normal field**
(Geodetic Reference System, GRS80)

**Spherical approximations**

**Spherical approximations + ellipsoidal corrections**

**Spherical approximations**
<table>
<thead>
<tr>
<th>Gravity field functionals</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Disturbing potential</strong></td>
</tr>
<tr>
<td>( T = V - U )</td>
</tr>
<tr>
<td>( T(\lambda, \vartheta, r) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^{n+1} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) )</td>
</tr>
<tr>
<td><strong>Gravity disturbances</strong></td>
</tr>
<tr>
<td>( \delta g = | \nabla T | \approx - \frac{\partial T}{\partial r} )</td>
</tr>
<tr>
<td>( \delta g(\lambda, \vartheta, r) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^{n+1} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) )</td>
</tr>
<tr>
<td><strong>Gravity anomalies</strong></td>
</tr>
<tr>
<td>( \Delta g = | g(r_r) - \gamma(r_r) | \approx - \frac{\partial T}{\partial r} - 2 \frac{T}{r} )</td>
</tr>
<tr>
<td>( \Delta g(\lambda, \vartheta, r) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^{n+1} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) )</td>
</tr>
<tr>
<td><strong>Surface density changes</strong></td>
</tr>
<tr>
<td>( \Delta \sigma \approx - \frac{1}{4\pi RG} \left( 2 \frac{\partial \Delta T}{\partial r} + \frac{\Delta T}{r} \right) \bigg</td>
</tr>
<tr>
<td>( \Delta \sigma(\lambda, \vartheta) = \frac{M}{4\pi R^2} \sum_{n=0}^{\infty} (2n+1) \sum_{m=-n}^{n} \Delta a_{nm} Y_{nm}(\lambda, \vartheta) )</td>
</tr>
<tr>
<td><strong>Total water storage changes</strong></td>
</tr>
<tr>
<td>( \Delta TWS(\lambda, \vartheta) = \frac{\rho_c R}{3} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Delta a_{nm} Y_{nm}(\lambda, \vartheta) )</td>
</tr>
</tbody>
</table>

---

Degree variances
Degree variances

Variance/Variability of geoid heights:

\[ \sigma^2 (N) = \frac{1}{4\pi} \int_{\Omega} N^2(\lambda, \vartheta) d\Omega \]

Variance of gravity disturbances:

\[ \sigma^2 (\delta g) = \frac{1}{4\pi} \int_{\Omega} \delta g^2(\lambda, \vartheta) d\Omega \]

In terms of spherical harmonics:

\[ \sigma^2 (N) = R^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} (C_{nm}^2 + S_{nm}^2) \]

In terms of spherical harmonics:

\[ \sigma^2 (\delta g) = \left( \frac{GM}{R^2} \right)^2 \sum_{n=0}^{\infty} (n+1)^2 \sum_{m=0}^{n} (C_{nm}^2 + S_{nm}^2) \]

Degree variances:

\[ \sigma_n^2 = \sum_{m=0}^{n} (C_{nm}^2 + S_{nm}^2) \]
Gravity disturbances

Geoid heights [mGal]

Upward continuation
Gravitational potential:

\[ V(\lambda, \vartheta, r) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^n \sum_{m=-n}^{n} a_{nm} Y_{nm}(\lambda, \vartheta) \]

---

**GRACE:**

\[ R = 6378 \text{ km} \]
\[ r = R + 450 \text{ km} \]

**Damping factors**

\( \left( \frac{R}{r} \right)^{n+1} \)
- \( n = 0 \) für \( 0 \) km
- \( n = 2 \) für \( 934095 \) km
- \( n = 120 \) für \( 0.000261 \) km

\( \left( \frac{R}{r} \right)^{n+1} \)
- \( n = 180 \) für \( 0.00004 \) km

---

Upward continuation
Degree variances

Satellite height (450 km):

: Earth surface (0 km)

Time variable part
Lecture: Satellite Altimetry
a 1.5 hour crash course

Wolfgang Bosch

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

What you shall learn:

Altimetry: how does it work?
Missions: which, when, what properties?
Resources: Where to get what data (& doc‘s)?
Data: organisation, content, format
Tools: read, extract, decode data
Sampling: spatio-temporal resolution; aliasing
Gridding: brute force and sophisticated SSHs
XO-Analysis: taking advantage of redundancy
Time series: analysis and interpretation
PCA: identify dominant SSH variability
DOT: the geodetic way to surface circulation
What you get

A memory stick with
- Slides of this (all) lessons
- Additional infos (glossary & abbreviations)
- Altimetry data (GDR-like and stacked)
- Portable version of Qtoctave (a Matlab clone with GUI)
- scripts for your exercises

Altimetry: how does it work?

- Most Altimeter Systems are realized by radar technology
- ICESat was carrying „GLAS“, a Geoscience Laser Altimeter

Typical Characteristics (Radar):

- Carrier frequency: 13.5 GHz
- Pulse duration: 12.5 nsec
- Pulse travel time: 5 msec
- Pulse repetition: 1000Hz
- Averaging: 0.05 sec
- Satellite height: 800km
- Radius of „footprint“: 2-11 km
- Ground velocity: 6.7 km/sec
Altimetry: how does it work?

Return signal & Analysis

Footprint development

Idealized echo (ocean surface)
(waveforms)

round trip travel time (half power point) ➔ height above sea level
slope of leading edge ➔ significant wave height
energy balance ➔ backscatter coefficient ➔ wind speed

Missions: which, when, what properties?

Mission (repeat cycle[days]/ground track separation at equator [km])

ERS1 was operated with different repeat cycles (3,35,169 days/?, 80, 16 km)
ERS2: tape recorder failed late 2007; since then data limited to direct downloads
Orbits of TOPEX-EM, Jason1-EM and EN were shifted to double/improve spatial resolution
ICESat: only episodic operations due to failure/problems of Laser
**Topex/Poseidon**

- Most sucessfull altimeter mission ever
- about 13 years operation
- 9,9516 repeat cycle
- Ground track distance 311km
- Precise orbits through Laser, DORIS and GPS
- First two frequency altimeter sensor
- Continuous calibration
- Latitude coverage ±66.0°

- Follow-on mission: Jason1

---

**ENVISAT**

- Operated by ESA
- Largest and most complex environmental satellite
- In operation since March 2002
- 35 day repeat cycle
- Ground track distance 80km
- Latitude range ± 81.5°
- Two frequency altimeter sensor
- Automatic tracking mode switching allows observation over lakes, rivers, ice, and land
CryoSat-2

Objectives
- thickness of land ice and sea ice
- melting of the polar ice
- Sea level rise
Launch Apr. 2010

Orbit
- Inclination I = 92°
- Mean height 717 km
- Repeat cycle 369 days (30 d sub cycle)
- 7.5 km track separation

Measurement modes
- Ku-band only, no radiometer
- LRM pulse limited
- Delay Doppler
- Interferometric SAR mode

SARAL/Altika

- Indian Space Research Organization (ISRO)
  - CNES: Altimeter
  - Alti-Ka
  - Ka-band 0.84 cm (viz 2.2 cm at Ku-band)
  - Bandwidth (480 MHz) => 0.31 p (viz 0.47)
  - Otherwise “conventional” RA
  - PRF ~ 4 kHz (viz 2 kHz at Ku-band)
  - Full waveform mode
- P/L includes dual-frequency radiometer
- Sun-synchronous, 35-day repeat cycle
- Navigation and control: DEM and DORIS
- Launch late 2010

Coastal relevance?
- Smaller (along-track) footprint than Ku-band RAs
- Longer repeat orbit
- Better SSH precision
- Soon to be operational
**HY-2A (China)**

- Chinese Academy of Sciences
- Ku- and C-band (conventional RAs)
- Payload also includes Scatterometer, 2-frequency radiometer, microwave imager
- Sun-synchronous orbit, 963 km, 99.3° inclination
- Repeat: 14 days (2 years) & 168 days (1 year) w/ 5-day sub-cycle
- Launch September 2010

*Coastal relevance?*
- Non-repeat orbit
- Soon to be operational

Note: this figure, usually claimed to be HY-2A, most likely is for its predecessor, HY-1

**Sentinel-3 (ESA)**

- European mission
- 7-year design lifetime (12 year reserves)
- Sun-synchronous, 27-day repeat cycle, inclination 98.65°
- Ku/C Radar Altimeter (SRAL)
- SAR (DDA) & conventional RA modes
  - DEM, multi-mode tracker, full-waveform subsets
- Dual Frequency Radiometer
- Payload includes Ocean and Land Color Instrument (OLCI), Sea and Land Surface temperature (SLST)
- Launch ~ 2012

*Coastal relevance?*
- Longer-repeat orbit
- SAR (DDA) mode
- Full waveform mode
- Operational w/in a few years
# Altimetry Missions – Main Characteristics

<table>
<thead>
<tr>
<th>Mission</th>
<th>Pulsewidth-limited altimeter systems</th>
<th>New technologies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Geosat&lt;sup&gt;1)&lt;/sup&gt;</td>
<td>ERS-1</td>
</tr>
<tr>
<td>Launch (month/year)</td>
<td>03/85</td>
<td>07/91</td>
</tr>
<tr>
<td>Acquisition until (month/year)</td>
<td>09/89</td>
<td>03/96</td>
</tr>
<tr>
<td>Mean height (km)</td>
<td>785.5</td>
<td>785.0</td>
</tr>
<tr>
<td>Inclination (°)</td>
<td>108.0</td>
<td>98.5</td>
</tr>
<tr>
<td>Latitude coverage (°)</td>
<td>±72.0</td>
<td>±81.5</td>
</tr>
<tr>
<td>Repeat cycle (days)</td>
<td>17.05</td>
<td>3/35/168</td>
</tr>
<tr>
<td>Track separation (km)</td>
<td>165&lt;sup&gt;6)&lt;/sup&gt;</td>
<td>933/30/16</td>
</tr>
<tr>
<td>Frequencies (GHz)</td>
<td>13.5</td>
<td>13.5</td>
</tr>
<tr>
<td>Altimeter noise (cm)</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>Radiometer/Frequencies</td>
<td>no</td>
<td>yes/2</td>
</tr>
</tbody>
</table>

1) Geosat had two different mission phases, a ‘geocentric mission’ (GM) with a non-repeat, drifting orbit, and an ‘exact repeat mission’ (ERM) with the orbit characteristics given in the table.

2) After the tandem configuration with Jason-1 (up to 08/2002) the TOPEX/Poseidon orbit was shifted by half the track separation to double the spatial resolution of both missions.

3) GFO continues to observe the same ground tracks as monitored by Geosat ERM (exact repeat mission).

4) Jason-1 continues to observe the same ground tracks as monitored by TOPEX/Poseidon until 08/2002.

5) Envisat continues to observe the same ground tracks as ERS-2.

6) SIRAL = Synthetic Aperture Interferometric Radar Altimeter.

7) A/A.

---

**Missions: which, when, what properties?**

**Ground track pattern of TOPEX (10 day repeat, 311km spacing), ENVISAT (35 day repeat, 80 km spacing)**
Orbit configuration - Rationale

- The energy limits of the radar system require satellite heights between 500 and 1500 km
- High orbits are less affected by air drag (TOPEX/Poseidon and Jason1 are at 1336km height others at ~800km heights)
- Small excentricities shall provide everywhere same precision
- Inclination determines the latitudal coverage

- Repeatability is choosen to get reliable time series over the same ground track
- Short repeatability and high spatial resolution exclude each other (due to orbit dynamics)
### Where to get what data?

<table>
<thead>
<tr>
<th>Mission</th>
<th>Cycle [days]</th>
<th>Provider</th>
<th>Access</th>
<th>Medium</th>
<th>Volume [GByte]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geosat</td>
<td>GM &amp; ERM(17)</td>
<td>NOAA</td>
<td>Free</td>
<td>CD-ROM</td>
<td>~ 6.5</td>
</tr>
<tr>
<td>ERS-1</td>
<td>3,35,168</td>
<td>ESA</td>
<td>Accepted proposal 1)</td>
<td>DVD, ftp</td>
<td>~ 30.0</td>
</tr>
<tr>
<td>TOPEX/Poseidon</td>
<td>9,9516</td>
<td>CNES/JPL</td>
<td>Free</td>
<td>CD-ROM DVD, ftp</td>
<td>~ 80.0</td>
</tr>
<tr>
<td>ERS-2</td>
<td>35</td>
<td>ESA</td>
<td>Accepted proposal 1)</td>
<td>DVD, ftp</td>
<td>~ 55.0</td>
</tr>
<tr>
<td>GFO</td>
<td>17</td>
<td>NOAA</td>
<td>Free</td>
<td>DVD, ftp</td>
<td>~ 35.0</td>
</tr>
<tr>
<td>Jason1</td>
<td>9,9516</td>
<td>CNES/JPL</td>
<td>Free</td>
<td>DVD, ftp</td>
<td>~ 20.0</td>
</tr>
<tr>
<td>ENVISAT</td>
<td>35</td>
<td>ESA</td>
<td>Accepted proposal 1)</td>
<td>DVD, ftp</td>
<td>~ 1200</td>
</tr>
</tbody>
</table>

1) Free access for scientific purpose if there is a project proposal accepted by ESA, see [http://eopi.esa.int/esa/esa/](http://eopi.esa.int/esa/esa/)

### Documents for altimeter mission data


- **ENVISAT:** [ENVISAT RA2/MWR Product Handbook](http://envisat.esa.int/pub/ESA_DOC/ENVISAT/RA2/ra2.ProductHandbook.1_2e.pdf.zip), Ed. 1.2, September 2004 (zipped pdf) see also [html-Version](http://envisat.esa.int/dataproducts/ra2/CNTR.htm)

- **GFO:** [GFO GDR User Handbook](http://ibis.grdl.noaa.gov/SAT/gfo/gdr_hbk.htm), June 2002
Providers of value-added altimeter data
(non-exclusive list!)

AVISO Archiving, Validation and Interpretation of Satellite Oceanographic data http://www.aviso.oceanobs.com

Provides access to

- GDR and IGDR mission data of Topex/Poseidon, Jason-1 (on behalf of CNES; binary format)
- CorSSH and SLA for most of the repeat missions (User handbook: DT CorSSH and DT SLA Product Handbook, CLS-DOS-NT-05.097) in a) gridded form or b) along-track in i) near real time or ii) delay time mode in netcdf-format
- Special products like Mean sea surface models (MSS) of (absolute) dynamic ocean topography (ADT or DOT)
- Wind & wave data (wind speed and significant wave height)
- Auxilary data

Access free but subscription required

CTOH (Centre of Topography of the Oceans and the Hydrosphere http://ctoh.legos.obs-mip.fr)

Provides (for scientific users):

- alongtrack GDR data with up-to-date corrections (for Topex/Poseidon, Jason-1, Jason-2, GFO, ENVISAT).
- coastal alongtrack GDR data with specific Xtrack processing
- global surface currents (Geostrophic and Ekman) from 1999-2008

Close cooperation with

- Hydroweb (for lake and river levels) and
- OSCAR (for ice products)
Providers of value-added altimeter data
(non-exclusive list!)

PO.DACC Physical Oceanography DAAC
(http://podaac.jpl.nasa.gov/JPL)

- TOPEX/Poseidon and Jason-1 mission data: GDR, IGDR, OGDR (binary coded; on behalf of NASA)
- TOPEX and Jason-1 Sea Surface Height Anomalies (ASCII header followed by binary data records)
- TOPEX and Jason-1 along-track gridded sea surface heights (ASCII header with binary coded data)
- Historical data sets from GEOS-3 and Geosat

RADS Radar Altimeter Data Base System
http://rads.tudelft.nl/rads/rads.shtml

- An altimeter data base establishing harmonized, validated and cross-calibrated sea level data, developed and maintained by DEOS (Earth Oriented Space Research of the Delft Technical University)
- For GEOSAT, ERS-1, TOPEX, Poseidon, ERS-2, GFO, Jason-1 ENVISAT, Jason-2
- Most recent orbits, geophysical corrections and models are applied
- User can extract data with user defined options.
Providers of value-added altimeter data
(non-exclusive list!)

ESA/CNES Basic Radar Altimeter Toolbox (BRAT)
http://earth.eo.esa.int/brat/html/data/toolbox_en.htm

Software Toolbox (v2.1.1, June 25, 2010)

- read all altimetry mission data for ERS-1/2, Topex/Poseidon, GFO, Jason-1, Envisat, Jason-2 and Cryosat, (from SGDR to gridded merged data)
- do some processing and computations
- visualise the results

Includes

- A tutorial on altimetry and a mission overview
- Description of applications

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Providers of value-added altimeter data
(non-exclusive list!)

DGFI OpenADB Open Altimeter Data Base
(http://openadb.dgfi.badw.de, experimental)

Similar intention like RADS (DEOS):

- altimeter data base with easy update capability for individual record parameter.
- Data extracts with number and sequence of record parameters defined by users.

Two data structures:

- MVA for along track data, and
- BINS for time series analysis (similar to NASA’s along-track gridded products)
Data: organisation, content, format

- „Level2“ or „Geophysical Data Records“ (GDR)
  Includes: precise orbit, range measurement, instrument status and health and all environmental and geophysical corrections

- Altimeter mission data is sequentially ordered and structured according to the following hierarchy:
  - Mission -- 1:n -- cycles -- 1:n -- passes

- Passes: duration during which the subsatellite track moves with either
  - increasing latitude "ascending pass", or
  - decreasing latitude "descending pass"

- Why this partitioning?
  - best suited for follow-on analysis of the data (repeat-pass and crossover)

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Data: organisation, content, format

- There is NO standard format! Every mission has its own format

The only (initial) convention is:
- Altimeter mission data is binary coded with parameter values stored as scaled integers, 4, 2, or 1 Byte in length
  e.g. latitude 126.2346° is resolved with 10^-6 degree; Thus scaled latitude is 126234600 and then stored as 4 Byte integer

Advantage:
- Compact storage
- Protection against accidental editing

Disadvantage:
- Can be read only by decoding software
(Tools to read binary coded data)

- Mission specific interface programs (usually in C and Fortran) provided by mission data providers – to be adapted by user requirements

- Binread: a generic C-program to decode and extract binary coded data. Requires a 'record map', describing sequence and coding of record parameters

- Matlab/octave functions: readrecmap.m & bin2dat.m to decode and load binary coded data. Requires a 'record map', describing sequence and coding of record parameters

- How does a 'record map' look like?

(What is a record map?)

- Record map: ASCII-file defining sequence and coding of record parameters of binary coded data

- Geosat.rmp

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<th>Description</th>
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Format more and more applied in altimetry: netcdf

- Self-explained freely available machine-independent format for binary storage of multi-dimensional data
- Developed by Unidata
- See [http://www.unidata.ucar.edu/software/netcdf](http://www.unidata.ucar.edu/software/netcdf)
- Libraries for C/C++ and Fortran with interfaces to MATLAB, Objective-C, Perl, Python, R, Ruby, Tcl/Tk

- Basic programs: ncgen, ncdump, and nccopy
- Matlab Interface: netcdf.m
- Java Browser: ncbrowse

Binary coded data of Jason1
prepared for an exercise to estimate ocean topography

Record map:

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<th>Value</th>
<th>Data Type</th>
<th>Description</th>
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../data/altimetry/jason1

L 101
L 101_001ssgh.00
L ...
L 101_254ssgh.00
L 102
Sample code to decode binary data
by means of matlab/octave functions
readrecmap.m and bin2dat.m

pfad = 'D: \data\altimetry\jason1'; % path to jason1 (to be adapted)
cycle = '101' % one particular cycle
recmap = 'ssgh.rmp' % record-map file binary coded data
format short

recmapfile = fullfile (pfad,recmap) % path to recordmap file
[byte,exps] = readrecmap (recmapfile) % read and get record structure

binfile = fullfile (pfad,cycle,'101_127ssgh.00') % path to a particular binfile
[data,nrec] = bin2dat (binfile,byte,exps); % decode all records of binfile

lonlatssh = data(:,1), data(:,2), data(:,4)]; % select lon, lat, & ssh param.

format long g
lonlatssh(1:5,:) % output first 5 records
lonlatssh(nrec-4:nrec,:) % output last 5 records

Sample output: decoding binary data
by means of readrecmap.m and bin2dat.m

>>>recmapfile = C:\Users\bosch\DATEN\data\jason1\ssgh.rmp
byte =
  4  4  4  4  4  4  4  4  2
exps =
  -6  -6  -5  -3  -3  -3  -3  -3
binfile = C:\Users\bosch\DATEN\data\jason1\101\101_127ssgh.00
ans =
  76.993642  -56.703619   28.708
  77.06203   -56.666922   28.747
  77.130277  -56.630182   27.248
  77.198383  -56.593399   28.363
  77.266349  -56.556575   27.591
ans =
  157.558032   59.643155   14.887
  157.639667   59.675656   14.876
  157.721467   59.7081     14.801
  157.967869   59.805092   14.635
  158.050337   59.837309   14.55
From data to sea surface heights

Necessary Information:
- Satellite position by precise orbit determination \(xyz\) \(\rightarrow\) ellipsoidal coordinates
- Altimeter range corrected for instrumental-, media-, and target-corrections
- \(\text{SSH} = \text{hsat} - \text{range}\)

Necessary (critical) corrections [order of magnitude]

- Orbit error (radial component) \(< 1 \text{ dm}\)

Instrumental effects
- Electronic time delay
- Clock (oscillator) drift
- Offset antenna phase centre
- Centre of gravity
- Time tagging of observations
- Doppler shift error

Atmospheric refraction (signal delay) due to
- Ionosphere \([\sim 3-5\text{ cm}]\)
- Troposphere, dry component \([\sim 2.3\text{ m} !!]\)
- Troposphere, wet component \([\sim 3-45\text{ cm}]\)

Target (ocean surface)
- Ocean tides [up to few m], loading effects [\(\sim 10\%\)]
- Earth tides \([\sim 3\text{ dm}], \text{pole tide} \sim 1\text{ cm}\)
- Electromagnetic bias (sea state) \([\sim 5\% \text{ of SWH}]\)
- Inverted barometer effect [up to 3 dm]
Wet troposphere correction – why is it critical?

Facts:
• Most modern altimeter systems carry on board radiometers.
• Radiometer observe brightness temperature (BT) at different channels
• The total water vapor content can be estimated by an empirical linear combination of BTs of different channels

Problems:
• The radiometer beamwidth causes a footprint radius of about 50 km
• The emissivity of ocean and land surfaces are very different
• BT observations become unreliable as soon as the satellite approaches the coast

Strategy:
• Take water vapor content from Met.-Services (ECMWF, NCEP, ...)
• Account for mixed land/ocean footprint (S.Brown algorithm)
• Estimate water vapor content from GPS observations

MSL: Global map of time averaged sea surface heights
Focus of MSL to the West Pacific

Mean Sea level is (to first order) in balance with gravity
Least Squares Harmonic Analysis

- **Estimating periodic oscillations of sea surface heights or sea level anomalies**

\[ h_q(\Delta t_k) + v_{qk} = c_q + d_q \cdot \Delta t_k + \sum_{p=1}^{2} A_{pq} \cdot \cos(\omega_p \cdot \Delta t_k - \Phi_{pq}) \]

where

- $\Delta t_k$: times of observation, relative to a reference epoch
- $h_{qk}$: observed sea surface height at point q and time $t_k$
- $v_{qk}$: sea surface height residual
- $c_q$: mean value of ssh at point q (solve-for parameter)
- $d_q$: drift term at q (solve-for parameter)
- $A_{pq}$: Amplituds of the p-th period at point q (solve-for parameter)
- $\Phi_{pq}$: Phases of the p-th period at point q (solve-for parameter)
- $\omega_p$: angular velocity for the p-th period $\omega_p = \frac{2\pi}{T_p}$, e.g. $T_1 = 365.25$, $T_2 = 182.625$

Time series of Topex SLA

for period 01/1993 – 07/2002 (Cycle 011 – 365)

```
.../data/altimetry/topex
L 011sla.xyz
L 012sla.xyz
L ...
L 364sla.xyz
L 365sla.xyz
```

The files `nnn.sla.xyz` give sea level anomalies SLA

SLA = instantaneous sea level w.r.t a long term mean sea level MSL [in this case CLS01]

for a sequence of 10-day TOPEX cycles `nnn = 011..365` (period 01/1993 – 07/2002)

The SLA were gridded from TOPEX ground tracks to a grid (which was adapted to the output of an ocean model and) whose grid nodes are defined by `lon = linspace (1.75, 359.875, 192); lat = linspace (-65.625, 65.625, 71);` Grid nodes over land are not listed. Every file contains exactly the same sequence and number grid nodes.
Code snippet for harmonic analysis of gridded SLA data

```matlab
pfad = 'C:\Users\bosch\DATEN\data\topex' ; \% pfad is to be adapted to stick
pattern = '.sla.xyz';
ftfile = 'fileliste'; \% list of sla files with their epochs

df = dir( fullfile (pfad,['*pattern '*']) ); \% get a handle to all sla files
nf = length(df); \% the number of sla files
R = []; for i =1:nf
    slafile = fullfile (pfad, df(i).name) \% a single sla file
    sla = load (slafile);
    R = [R sla(:,3)]; \% concatenate sla to an (n x q) matrix R
end
[n q] = size(R)
data = dlmread (fullfile (pfad, ftfile)) \% get file list with associated epochs
dtimes = data(:,2)
cs = [ones(q,1) dtimes]; \% initialize Jacobi matrix for mean and drift
for k=1:1
    w = 2*pi/365.25;
    cs = [cs, cos(w*dtimes), sin(w*dtimes)]; \% Jacobi matrix
end
X = ((cs'*cs) \ (cs'*R'))' \% least squares estimate of mean, drift, cos, and
sin-term
```

Annual sea level variability

Annual signal explains about 22% of observed sea level anomalies

Sin - term (1: April & -1: October)

Cos - term (1: January & -1: July)
Secular sea level change for the decade 08/1992 – 08/2002

The sampling problem

Space and time scales of ocean processes (© D. Chelton)

Visible by single altimeter satellites:

Envisat
Topex/Poseidon
The alias problem

Whenever high-frequency signals are sampled with rather long period, then the high frequency signal appears with a period which is even much longer than the sampling period.

E.g.: M2 tide is a semi-diurnal sea level change
if sampled by Topex (every 9.9156 days) it maps into a 62.1 day alias period

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<th>S₂</th>
<th>N₂</th>
<th>K₂</th>
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Table: Alias and Rayleigh periods [days] for ERS-2/ENVISAT

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**Space-time sampling by multi-mission altimetry**

- **Topex/Jason1 & Topex-EM**
  10 day repeat;
  ~2.8° (1.4°) eq. track distance

- **GFO**
  17 day repeat;
  ~1.6° eq. track distance

- **ERS1/2 & Envisat**
  35 day repeat;
  ~0.8° eq. track distance

- **ERS-1 & Geosat geodetic phases**
  (not shown)
Identifying aperiodic sea level variation

**Principal Component Analysis (PCA)**

**or**

**Empirical Orthogonal Functions (EOF) Analysis**

**What is it for?**
- For a given multi-variate random time series PCA identifies those spatial pattern (eigenvectors, EOF’s) and their temporal evolution (principal components, PC) that explain – in decreasing order – the most dominant contributions to the signal variance.

“Mode”: A single eigenvector (EOF) and the associated PC’s
- The first mode explains the most dominant part of the signal
- The second mode explains the second largest signal contribution
- Allows to control the degree of approximation
- PCA is the most economic representation of a multivariate time series
Sea level anomalies as multi-variate time series

Observed:
Sea level anomalies at $n$ locations for $q$ epochs

Assumption:
Data is centered: temporal mean subtracted

Re-arranged:
Residual matrix $Y$ with size $(n \times q)$

Eigenvalue analysis of the $(n \times n)$ signal covariance $C = Y \cdot Y'$

$C$ is a positive-semidefinite quadratic form, such that $\lambda_j \geq 0$

---

Code snippet for PCA of gridded SLA data

```matlab
pfad = 'D:\data\altimetry\topex';       \% path is to be adapted to stick
pattern = '.sla.xyz';
ftfile = 'fileliste';                    \% list of sla files with their epochs

[n q] = size(R)                           \% R-matrix as in previous snippet
data = dlmread (fullfile (pfad, ftfile)) \% get file list with associated epochs
dtimes = data(:,2)
rmean = mean(R)';                           \% find mean values w.r.t time
R = R - rmean*ones(1,q);                   \% perform residuals w.r.t. mean values

[u,s,v] = svd (R,xy);                     \% perform singular value decomposition
lambdas = diag(s(1:q,1:q)^2);            \% eigen values are singular values squared
evs = [ xy(:,1:2), u(:,1:q)]            \% eigen vectors (=spatial pattern)
A = (u(:,1:q))'*R                         \% Principle components (= temporal coefficients)
pcs = [dtimes', A']                        \% plotting not included
```

---

Here: seasonal signal (annual + semi-annual period) removed

El-Niño pattern

annual signal

La Niña pattern
PCA – a tool to identify data problems

Mode 6 with a high frequency of the principle components and a track-related spatial pattern!

Fourier-Analysis of PCs shows:
- Period = 62 days
- „Alias period“ if M2 (12 hours) is sampled by TOPEX every 9.9156 days

Obviously a significant error of the tide model applied to correct SSH

Dynamic Ocean Topography (DOT)

Hydrodynamic processes (density differences, wind pressure) cause the sea level to deviate from a geopotential surface (geoid). This deviation is called Dynamic Ocean Topography (DOT). It has a magnitude of only ±1–2 metres.

There are two independent ways to assess the DOT
a) model the hydrodynamic processes
b) estimate the difference between sea level and geoid: DOT = SSH - N
DOT: The geodetic way

Equation

\[ \text{DOT} = \text{SSH} - \text{N} \]

not as simple as it seems!

Geoid heights \( \text{N} \)
- are defined everywhere
- Relative smooth spherical harmonics)

Sea surface heights \( \text{SSH} \)
- observed on ocean profiles only
- High along-track resolution (e.g. 7km sampling)

Strategies to perform DOT = SSH – N with consistent filtering

Global approach
- rationale: filtering and performing differences is easy in the spectral domain. Geoid is already defined in terms of spherical harmonics. Thus SSH are expanded into spherical harmonics.
- Problem: SSH over land undefined! How to handle this? Fill land area with geoid. Step function at the coast remains and must be smoothed.

Profile approach
- Rationale: stay as long as possible on the altimeter ground tracks to maintain the high resolution and avoid undesirable gridding of SSH
- Problem: Systematic differences if SSH is filtered along track (1-D) and the geoid is filtered spectrally (2-D) – can be accounted for by a „filter correction“
Data profiles of Jason1 (binary coded)
prepared for an exercise to estimate ocean topography

Record map:

<table>
<thead>
<tr>
<th></th>
<th>9</th>
<th>34</th>
<th>ssgh.rmp</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>9</td>
<td>34</td>
<td>ssgh.rmp</td>
</tr>
<tr>
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<td>+4</td>
<td>-6.deg</td>
<td>glon.00</td>
</tr>
<tr>
<td>002</td>
<td>4</td>
<td>-6.deg</td>
<td>glat.00</td>
</tr>
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<td>4</td>
<td>-5.d</td>
<td>jday.00</td>
</tr>
<tr>
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<td>4</td>
<td>-3.m</td>
<td>ssh.03</td>
</tr>
<tr>
<td>005</td>
<td>4</td>
<td>-3.m</td>
<td>geoh.15</td>
</tr>
<tr>
<td>006</td>
<td>4</td>
<td>-3.m</td>
<td>geoh.00</td>
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<td>4</td>
<td>-3.m</td>
<td>sshs.08</td>
</tr>
<tr>
<td>008</td>
<td>4</td>
<td>-3.m</td>
<td>geohs.08</td>
</tr>
<tr>
<td>009</td>
<td>2</td>
<td>-3.m</td>
<td>dot.18</td>
</tr>
</tbody>
</table>

longitude of satellite footprint
geodetic latitude of satellite footprint
julian day epoch 2000.0
sea surface heights (unfiltered)
geoid heights ITG-Grace03s (satellite-only, unfiltered)
geoid heights EGM2008 (high resolution, unfiltered)
smoother sea surface (Gauss filter length D = 97 km)
smoother geoid heights (GOCO02S; Gauss filter length D = 97 km)
dynamic ocean topography (DOT); DGFI-version

.../data/altimetry/jason1

L 101
<table>
<thead>
<tr>
<th></th>
<th>101_001ssgh.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>...</td>
</tr>
<tr>
<td>L</td>
<td>101_254ssgh.00</td>
</tr>
</tbody>
</table>

Code snippet to estimate DOT on individual profiles

pfad = 'D: ??? \data\altimetry\jason1'; % path to jason1 (to be adapted)
cycle = '101'; % one particular cycle
citmap = 'ssgh.rmp'; % record-map file binary coded data
recmapfile = fullfile (pfad,citmap) % path to recordmap file
[byte,exp] = readrecmap (recmapfile) % read and get record structure
dir = dir(fullfile(pfad,citmap)); % list of files in the cycle directory
xyz = [];
for i = 1:length(df) % loop to run through all pass-files
    binfile = fullfile(pfad,citmap,df(i).name) % construct full path to binfile
    [data,nrec] = bin2dat (binfile,byte,exp); % decode all data of a pass
    xyz = [xyz; data]; % concatenate all data from all passes to xyz
End % xyz available as dgfi_data.mat for ocean Exercise

% uncomment only one of the following lines (according to record map ssgh.rmp)
% DOT = xyz(:,4) - xyz(:,5); % DOT = SSH - N (unfiltered with ITG03S-geoid)
DOT = xyz(:,7) - xyz(:,8); % DOT = SSH - N (filtered with GOCO02S)
% DOT = xyz(:,9); % DOT (with filter correction); DGFI version
% gridding (not shown here)
10-day DOT snapshot (Jason1 and Topex)

Mean GOCE DOTs (D=121km/L=120)
compared with external estimates
DOT evolution South Atlantic / Agulhas Stream
Multi-mission altimetry – GOCO02S geoid (Filter with D = 70km)

Geostrophic velocities for South Atlantic / Agulhas Stream
## Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>SLA</td>
<td>Sea level anomaly</td>
</tr>
<tr>
<td>DOT</td>
<td>Dynamic Ocean Topography</td>
</tr>
<tr>
<td>SSH</td>
<td>Sea surface height</td>
</tr>
<tr>
<td>GDR</td>
<td>Geophysical data record</td>
</tr>
<tr>
<td>SWH</td>
<td>Significant waveheight</td>
</tr>
<tr>
<td>MWR</td>
<td>Microwave radiometer</td>
</tr>
<tr>
<td>GLAS</td>
<td>Geoscience Laser Altimeter (on ICESat)</td>
</tr>
<tr>
<td>PRF</td>
<td>Pulse repetition frequency</td>
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</tbody>
</table>
Lecture: Analysis Tools

Jürgen Kusche

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

Analysis Tools

Why this lecture?

By now (see lecture by T. Mayer-Gürr and F. Flechtner) it has become clear that GRACE solutions (say SH coefficients converted to TWS, total water storage) require some post-processing by the user, beyond projecting the coefficients into space domain

- to suppress correlated noise, remove ’stripes’ (→ filtering)
- to extract the dominating ’modes’ of temporal variability (→ PCA, ...)

Being in general use, these analysis tools always remove signal content together with ’noise’. For any comparison of GRACE data with geophysical modelling, it is imperative therefore that the same tool is applied to both. For getting ’absolute’ amplitudes, rates, etc., it is imperative to consider the ’bias’ of an analysis technique.
This lecture will deal with both filtering techniques for data on the sphere and with PCA-related techniques.

While the first is often considered an obscure magic conceived by geodesists, the second is in general use in the atmosphere/ocean communities. See textbooks by Preisendorfer, Joliffe, von Storch & Zwiers...

Note: If you download GRACE gridded products from GRACE Tellus website, GFZ ICGEM or others, some filtering has been applied already.

Analysis Chain: Filtering

1. **GRACE SHC**
   - Treatment degree 1 & 2, temporal anomalies wrt epoch t, restore AOD
2. **Filtering (in spectral domain)**
3. **Mapping to space domain or basin averaging**
4. **OR Filtering (in space domain)**

5. **SHA geophysical model → SHC**
   - Temporal alignment, possibly remove deg. 1 consistent to GRACE
2. **Filtering (in spectral domain)**
3. **Mapping to space domain or basin averaging**
4. **OR Filtering (in space domain)**

Globally defined?

Multiplication in spectral domain

Convolution on the sphere
Analysis Chain: Filtering - quite often

GRACE SHC $\tilde{\epsilon}_{nm}$, $\tilde{s}_{nm}$ → Geophysical model

Treatment degree 1 & 2, temporal anomalies wrt epoch t, restore AOD → Temporal alignment, possibly remove deg. 1 consistent to GRACE

Filtering (in spectral domain) → Filtering (in spectral domain)

Mapping to space domain or basin averaging

Filtering (in space domain)

Analysis Tools: PCA

Filtering (in spectral domain) → Filtering (in spectral domain)

Mapping to space domain or basin averaging

Computation of EOFs from GRACE or model

Project gridded data onto EOFs: PCs for GRACE and model

Truncated (filtered) reconstruction
Analysis Chain: PCA - also possible...

GRACE SHC $\tilde{c}_{nm}$, $\tilde{s}_{n|m}$ → SHA geophysical model $\rightarrow$ SHC

Treatment degree 1 & 2, temporal anomalies wrt epoch $t$, restore AOD

Temporal alignment, possibly remove deg. 1 consistent to GRACE

PCA & truncated reconstruction (in spectral domain)

Mapping to space domain or basin averaging

Part I: Filtering techniques and their application to GRACE data

Filtering

- Attempts to suppress ‘noise’ in data (here: SH coefficients)
- Requires that we have an a-priori knowledge of expected ‘noise’ (→ characterize spectral behaviour of noise)
- Filters take this into account either implicitly (‘deterministic filters’ or explicitly, ‘stochastic filters’)
- Also regularizing or ‘constraining’ GRACE normal equations corresponds to some kind of filtering (Kusche 2007, Klees et al 2008, Swenson & Wahr 2011)

Data sets e.g.

- GRACE-derived SH coefficients or maps of geoid, gravity anomalies, TWS
- SH coefficients or maps of TWS and other data from model output
What is the purpose of filtering a GRACE solution?

Unfiltered GRACE solution

Boxcar-filtered GRACE solution: Remove stripes by averaging, convolution
What is the purpose of filtering a GRACE solution?

Boxcar-filtered GRACE solution: Remove stripes by averaging, convolution
What is the purpose of filtering a GRACE solution?

Gaussian-filtered GRACE solution: convolution with a smooth kernel

\[ W(\lambda, \theta, \lambda', \theta') \]

\[ W(\psi) \]

Filtering a GRACE solution in spatial and in spectral domain

\[ F_W(\lambda, \theta) = \int_{\Omega} W(\lambda, \theta, \lambda', \theta') F(\lambda', \theta') d\omega \]

\[ \tilde{f}_{nm} = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \tilde{u}_{nm} \tilde{f}_{n'm'} \]
Part I: Filtering techniques and their application to GRACE data

\[ V(r, \theta, \lambda, t) = \frac{GM}{r} \]
\[ + \frac{GM}{r} \sum_{n=2}^{\infty} \left( \frac{R}{r} \right)^n \sum_{m=0}^{n} P_{nm}(\cos \theta) (C_{nm}(t) \cos m\lambda + S_{nm}(t) \sin m\lambda) \]

Notations I use
- Positive and negative order
- Fully normalized
- All factors are put into the coefficients

\[ \tilde{v}_{nm} = \bar{C}_{nm} \quad \text{for } m \geq 0 \]
\[ \tilde{v}_{nm} = \bar{S}_{n|m|} \quad \text{for } m < 0 \]

Potential → Surface Mass
\[ F = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{f}_{nm} \tilde{Y}_{nm}(\lambda, \theta) \]
\[ \bar{f}_{nm}(t) = R \frac{2n + 1}{3} \frac{\rho_c}{1 + k_n^2} (\bar{v}_{nm}(t) - \bar{v}_{nm}) \]

Part I: Filtering techniques and their application to GRACE data

Isotropic filters
\[ F_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{f}_{nm} \tilde{Y}(\lambda, \theta) \]
\[ \bar{f}_{nm} = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \bar{\omega}_{nm} \bar{f}_{n'm'} \]
\[ \bar{\omega}_{nm} \bar{f}_{n'm'} = \delta_{nm} \bar{f}_{n'm'} \]
\[ F_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n \bar{f}_{nm} \tilde{Y}(\lambda, \theta) \]
Part I: Filtering techniques and their application to GRACE data

Isotropic filters: Gaussian filter (Wahr et al., JGR 1998)

\[ W_d(\psi) = \frac{2b}{1-e^{-2d}} \sum_{n=0}^{\infty} (2n+1)w^{(d)}_n P_n(\cos \psi) \]

\[ u^{(d)}_{n+1} = \frac{2n+1}{b} w^{(d)}_n + w^{(d)}_{n-1} \]

Destriping filters

\[ F_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tilde{f}_{nm}^W \tilde{Y}(\lambda, \theta) \]

\[ \tilde{f}_{nm}^W = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} a_{nm'}^n \tilde{f}_{nm'} \]

\[ a_{nm'}^n = \delta_{mm'} w_n \]

Isotropic | Non-isotropic
Part I: Filtering techniques and their application to GRACE data

Destriping filters

\[ W(\lambda, \theta, \lambda', \theta') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=-n'}^{n'} \sum_{m'=-n'}^{n'} \tilde{w}_{nm}^{n'm'} \tilde{Y}_{n, m}(\lambda, \theta) \tilde{Y}_{n', m'}(\lambda', \theta') \]

Swenson & Wahr
GRL, 2006

Figure 2. Stokes coefficients (C_m), converted to mass, plotted as functions of degree for orders m = 0–3 and m = 15–18. (a) and (c) Coefficients plotted for every degree; (b) and (d) coefficients plotted separately for even and odd degrees.
Part I: Filtering techniques and their application to GRACE data

Destriping filters and WRMS reduction (Kusche et al., J Geodesy 2009)

GRACE TWS WRMS [cm], 3 filters

<table>
<thead>
<tr>
<th>GRACE WRMS</th>
<th>DDK1</th>
<th>DDK2</th>
<th>DDK3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global</td>
<td>3.87</td>
<td>4.77</td>
<td>6.46</td>
</tr>
<tr>
<td>Continents</td>
<td>5.89</td>
<td>7.03</td>
<td>8.58</td>
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<td>5.33</td>
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<tr>
<td>Amazon</td>
<td>14.34</td>
<td>16.47</td>
<td>17.66</td>
</tr>
<tr>
<td>Sahara</td>
<td>1.75</td>
<td>2.56</td>
<td>4.82</td>
</tr>
</tbody>
</table>

stronger filtering less

Part I: Filtering techniques and their application to GRACE data

Basin averaging

Ocean mass

Jürgen Kusche

DFG SPP 1257
Part I: Filtering techniques and their application to GRACE data

Basin averaging

\[ \tilde{F}_O = \frac{1}{O} \int_O F d\omega = \frac{1}{O} \int \Omega F d\omega \quad \text{(spatial domain)} \]

\[ O = O(\lambda, \theta) = \begin{cases} 1 & (\lambda, \theta) \in O \\ 0 & (\lambda, \theta) \notin O \end{cases} \]

\[ \tilde{O} = \int \Omega Y_{00} d\omega = 4\pi \tilde{O}_{00} \]

\[ \tilde{F}_O = \frac{1}{O_{00}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tilde{O}_{nm} \tilde{f}_{nm} \quad \text{(spectral domain)} \]

Note: Equality requires

- the integral is discretized at an error small enough
- the SH truncation degree is big enough OR BOTH F and O are band-limited
Part I: Filtering techniques and their application to GRACE data

Smoothed basin averaging and bias

\[ \mathcal{O}_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathcal{O}_{nm}^W Y_{nm}(\lambda, \theta) \]

\[ \bar{F}_{OW} = \frac{1}{\mathcal{O}_{W}(\theta)} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n \mathcal{O}_{nm} \bar{f}_{nm} \]

\[ \beta_{\mathcal{O}, W, F} = \frac{\bar{F}_{OW}}{\bar{F}_O} = \frac{1}{w_0} \frac{\sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n \mathcal{O}_{nm} \bar{f}_{nm}}{\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathcal{O}_{nm} \bar{f}_{nm}} \]

Part I: Filtering techniques and their application to GRACE data

Leakage problem and filter bias: basin averaging, what happens?

Model

---

GRACE (truncated SH)
Part I: Filtering techniques and their application to GRACE data

Leakage problem and filter bias: basin averaging, what happens?

Model

GRACE (truncated SH)

+ noise

Leakage problem and filter bias: basin averaging, what happens?

Model

GRACE (truncated SH)

Exact averaging

Spectral leakage

Lost signal
Part I: Filtering techniques and their application to GRACE data

Leakage problem and filter bias: basin averaging, what happens?

Model

Lost signal (leaking out)

GRACE (truncated SH)

Bias can be computed from model

Leaking out

Leaking out

Lost signal (leaking out)

Signal added by surrounding region (leaking in)
Empirical orthogonal function analysis

Principle components

Related concepts

(Fig.: SST, from IPPC 4AR)

Part II: Principle component analysis and related ideas

PCA

- attempts to find a relatively small number of independent modes in a data set that convey as much as possible information without redundancy

- can be used to explore the structure of the data variability in an objective way, i.e. without assumptions on periodic behaviour etc.

- and EOF analysis are the same.

Data sets e.g.

- GRACE-derived maps of TWS, TWS and other maps from model output

- Sea level anomalies

- Other related spatial fields (SST, SLP, ...)

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Part II: Principle component analysis and related ideas

What does PCA do?

PCA uses a set of orthogonal functions (EOFs) to represent a spatio-temporal data field in the following way:

\[ y_i = \begin{pmatrix} y_{1;i} \\ y_{2:i} \\ \vdots \\ y_{n;i} \end{pmatrix} \]

epochs (time) \[
\begin{array}{c}
\downarrow \\
\sum_{j=1}^{n} d_{j;i} e_j = E d_i \\
\end{array}
\]

PCs (expansion coefficients)

EOFs (new basis)

EOF \( e_j = e(\lambda_j, 0) \) show spatial patterns of the major factors (‘modes’) that account for temporal variations.

PC \( d_{j;i} = d(t) \) tells how the amplitude of EOF varies with time.

How do we obtain the PCs?

Practical: WGHM and GRACE

\[ y_i = \sum_{j=1}^{n} d_{j;i} e_j = E d_i \]

\[ Y = (y_1, y_2, \ldots, y_p) = ED \]

Lets assume the EOFs are orthogonal (why are they called EOF, after all) and normalized.

\[ d_i = (E^T E)^{-1} E^T y_i = E^T y_i \]

\[ d_{j;i} = e_j^T y_i \]

The PCs are found as an orthogonal projection of the data onto the new basis functions (the EOFs). We can try to reconstruct the original data using only the ‘major’ EOFs.

\[ \bar{y}_i = \sum_{j=1}^{n} d_{j;i} e_j \]
Part II: Principle component analysis and related ideas

What do we get from PCA?

- $PC_1$ $t$ 62%
- $PC_2$ $t$ 24%
- $PC_n$ $t$ <1%

Example (I)
GRACE TWS 400km Gaussian filtered (note: EOF $\times$ PC has unit of data)
How do we choose the EOFs?

Total variance

\[ \Delta^2 = \text{trace} \left( EDD^T E^T \right) = \text{trace} \left( D^T D^T \right) = \sum_{j=1}^{n} \sum_{i=1}^{p} d_{j,i}^2 \]

Then, the maximum variance is concentrated in a single EOF if

\[ \mathbf{Y}^e \rightarrow \text{max. variance} \]

i.e., maximize \( \frac{1}{p} (\mathbf{Y}^e)^T (\mathbf{Y}^e) = \frac{1}{p} \mathbf{e}^T \mathbf{C} \mathbf{e} \quad \text{subject to} \quad \mathbf{e}^T \mathbf{e} = 1 \)

Or, solve an eigenvalue problem

\[ \mathbf{C} \mathbf{e} = \lambda \mathbf{e} \]

Some remarks (I)

- The covariance as above is temporal, i.e. it considers auto- and covariances of time series per grid point (for n grid points)

  (we’ll come back to this point)
The EOFs are found as the eigenvectors of the data covariance matrix.

Some remarks (II)

- It is an empirical realization. Should we know the true covariance, we might better use this one instead of the empirical one
- All data has been considered as (perfectly) centered
- Eigenvectors require normalization and a further convention for uniqueness, e.g.

\[ e_j^T e_j = 1 \]
\[ e_{1:j} > 0 \]
Part II: Principle component analysis and related ideas

The EOFs are found as the eigenvectors of the data covariance matrix.

Some remarks (III)

– Alternatively, we could use SVD applied to the data matrix

– Each EOF explains a fraction of the total variance, given by the ratio of the EV vs. TV (total variance)

\[ \Delta^2 = \frac{1}{p} \sum_{j=1}^{n} \left( \sum_{i=1}^{p} y_{j;i}^2 \right) = \text{trace}(C) \]

\[ \eta_j = \frac{\lambda_j}{\Delta^2} \]

– It is common to choose the number of EOFs as to „explain” 90% (or ...) of the TV

Example

GRACE global analysis (left: EV, right: cumulative percentage of TV)

(courtesy E. Forootan)
Part II: Principle component analysis and related ideas

The EOFs are found as the eigenvectors of the data covariance matrix.

Some remarks (IV)

- Instead of computing the temporal data covariance, we may compute the spatial covariance (spatial variance and covariances for the $p$ epochs)

$$
C' = \frac{1}{n} Y^T Y = \frac{1}{n} \begin{pmatrix}
\sum_{j=1}^{n} y_{j;1}^2 & \sum_{j=1}^{n} y_{j;1} y_{j;2} & \cdots & \sum_{j=1}^{n} y_{j;1} y_{j;p} \\
\sum_{j=1}^{n} y_{j;2} y_{j;1} & \sum_{j=1}^{n} y_{j;2}^2 & \cdots & \sum_{j=1}^{n} y_{j;2} y_{j;p} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{j=1}^{n} y_{j;p} y_{j;1} & \sum_{j=1}^{n} y_{j;p} y_{j;2} & \cdots & \sum_{j=1}^{n} y_{j;p}^2
\end{pmatrix}
$$

- Requires less memory for $p < n$
- Temporal and spatial covariance matrices share the same $p$ EVs
- $k$-th EOF (spatial) $\sim k$-th PC (temporal) and vice versa

Some remarks (V)

- Linear transformations of the data lead to new eigenvectors and $\lambda$-values

$$
z_i = A y_i, \quad i = 1 \ldots p
$$

$$
\lambda_i \in [\min(\mu_i) \cdot \min(\lambda_i), \max(\mu_i) \cdot \max(\lambda_i)]
$$

Therefore:

- EOFs and PCs computed on a regional grid look different from EOFs/PCs computed on a global grid
- GRACE: EOFs of geoid change look different from EOF of TWS change
- PCA applied to GRACE SH coefficients (EOF filter of Schrama & Wouters) looks different from PCA applied to grids.
Example (I) revisited

Annual signals out of phase

Semi-annual (modulation of annual)

Trend (decrease)+ annual

Trend ?

Trend (increase)+ annual

2007: unusually strong rainfall in Congo

2 rainfall pattern (Kasei, 2009)

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Part II: Principle component analysis and related ideas

Examples (II)
Wouters & Schrama (GRL, 2007) Direct EOF filtering of GRACE SH coefficients

Top: Unfiltered/Gaussian, Middle: EOF filtered, Bottom: difference. Unit [cm]

How many modes (EOFs) should we retain? In other words, how many % of the data TV should we reconstruct?

North et al. 1982, Month. Weath. Rev.: Consider the spatio-temporal data as stochastic, i.e. perturbed by e.g. Gaussian noise. Then, the covariance

\[ C = \frac{1}{p} YY^T \]

and the eigenvalues / -vectors will be stochastic as well. In first order...

\[ \delta \lambda_j = \sqrt{\frac{2}{n}} \lambda_j + \cdots \quad \delta \theta_j = \frac{\delta \lambda_j}{\lambda_k - \lambda_j} \theta_k + \cdots \]

If the sampling error in the eigenvalue is comparable to the spacing of the eigenvalues, then the sampling error of the EOF will be comparable to the nearby EOF.

And then it is time to truncate.
Comparing two (or multiple) data sets (e.g. GRACE TWS and hydrology TWS, altimetric sea level and model steric sea level)

1. If you trust all data are ‘consistent’, i.e. they show the same physics apart from sampling errors

\[ X = \left( Y^{(1)}, Y^{(2)}, \ldots, Y^{(m)} \right) \]

\[ C = \frac{1}{pM} XX^T \]

2. If not, use the same basis for comparing. Compute EOFs from the above or from one of the data sets, project data on these and compare PCs

\[ d_i^{(m)} = E^{(m)}_i y_i^{(m)} \quad \text{or} \quad d_i^{(m)} = E^{(m^*)}_i y_i^{(m)} \]
Part II: Principle component analysis and related ideas

Related concepts: (Orthogonal) EOF rotation

\[ y_i = \sum_{j=1}^{n} d_{j;i} e_j = Ed_i \]

\[ V^T V = I \]

\[ F = EV^T \]

\[ F^T F = V E^T E V^T = I \]

\[ y_i = F^T r_i \]

\[ r_i = V d_i = F^T y_i \]

\[ C_r = V A V^T \]

REOFs are still orthogonal - RPCs are correlated now.

How can we use this degree of freedom?

Part II: Principle component analysis and related ideas

Related concepts: REOF, how can we use this degree of freedom? I.e. how do we find the rotation matrix?

E.g. VARIMAX, maximize the variance of the square of the REOFs (i.e. the spreading of the total variability of the modes)

\[ F(V) = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} f_{j;i}^4 - \frac{\gamma}{n} \left( \sum_{j=1}^{n} f_{j;i}^2 \right)^2 \right) \]

\[ F = EV^T \]

E.g. ICA, minimize 3th / 4th statistical moments of the REOFs to maximize the independence of the RPCs
Take-home message

Filtering is a necessary tool for interpreting GRACE data correctly.

- What we have discussed here are the options that the user of Level-2 data products has. Some filtering has been applied in the Level-1 processing as well. There is simply no point in using ‘unfiltered’ data.

PCA is a useful tool for interpreting GRACE and other geophysical data and model outputs. It is either applied as a kind of filtering (see above) or as a tool to explore the major directions of data variability.

- It is easy to construct counterexamples where PCA fails to isolate physical modes!
- “PCA may help you to find the needle in the haystack. But once you found it, you should be able to recognize it as a needle“ [v. Storch]. I.e. you should be able to assign some physics to it, otherwise it might be just an artefact.
The global climate system and the water cycle

Andreas Hense

Sommerschule "Globaler Wasserkreislauf" 13. September 2011

Climate System as the bio-geophysical/chemical Earth system
The division is based on

- the media (Gases, water in liquid/solid phase, biological material porous solids)
- the time scales e.g. defined by the impulse response function / correlation function of typical fluctuation
- the couplings between the subsystems.

<table>
<thead>
<tr>
<th>mass $M$:</th>
<th>Atmosphere:</th>
<th>Ocean:</th>
<th>Lithosphere</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>16</td>
<td>0.5</td>
</tr>
<tr>
<td>heat capacity $M \cdot c_p$:</td>
<td>1</td>
<td>68</td>
<td>0.5</td>
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</tbody>
</table>

Interacting cycles

- Energy cycle: basic mathematical description is through the First Law of Thermodynamics
- Hydrological cycle: basic mathematical description is through the conservation law of water mass, continuity equation
- angular momentum cycle; basic mathematical description is through the conservation law of angular momentum (modified Newton's law, Navier Stokes equations in a rotating frame of reference)
trace elements and trace particles; basic mathematical description is mass conservation (continuity equation) and the law of mass action from chemistry e.g. for C, N, S (bio-geochemical cycles)

- the (non-reactive) bulk mass of each subsystem, mathematical description by mass conservation, continuity equation

Application of classical hydro-thermodynamics with phase changes of water and possibly reactive cycles, liquid/solid water and trace particles distinguished by particle size and shape

Boundary conditions

- **solar insolation** as external and variable parameter of the energy cycle
- **Land-Ocean distribution** e.g. influences the hydrological cycle
- **orography on land and in the ocean, land-ocean distribution und Earth rotation velocity and the Moon by its ephemerides** are relevant external parameter of the angular momentum cycle in atmosphere and ocean
- solar insolation influences bio-geochemical cycles e.g. ozone or through photosynthesis
Two definitions of the climate / Earth system which are complementary

- e.g. the energy cycle: First law $\Delta E = \delta q + \delta a$ in its variants for the subsystems
- the exchange of energy, momentum and mass across the boundaries between the subsystems: the interface between two systems cannot store a property therefore fluxes of energy, momentum and mass directed to or away from the boundary have to sum to zero: budget conditions

Further ordering is necessary to keep the problem tractable ...

climatology, scales in space and time, statistics and statistical physics
Spatial and temporal scales

- climate dynamics are realized on vastly different space and time scales
- for analysis one has to select a-priori specific scales
  - urban climate
  - regional or mesoscale climate
  - global climate

The problem of high dimensionality

- total volume of the global climate system: spherical shell of depth \( h \sim 100 \text{km} \)
- smallest unit volume \( V_0 \sim 1 \text{mm}^3 = 10^{-9} \text{m}^3 \) "Kolmogorov" micro-scale
- within each unit volume there are \( n_i \sim O(100) \) variables like \( T, p, \rho_i, u, v, \) etc

\[
V \sim 4\pi h R_{\text{Earth}}^2 \\
\sim 4\pi \times 10^5 (6.372 \times 10^{12}) \\
\sim 5 \times 10^{19} \text{m}^3
\]

\[
\frac{V}{V_0} \sim 5 \times 10^{28}
\]

\[
\frac{Vn_i}{V_0} \sim 10^{31}
\]

There are about \( 10^{31} \) degree-of-freedom to be treated
The Basics
The Energy and Water cycles

- arrange all degrees-of-freedom in a state vector $\mathbf{Y}$
- the classical statistical physics argument: detailed computations are not possible
- the climate state vector is a random variable $\mathbf{Y}$
- define a probability density $p(\mathbf{y})$ for a specific climate state

$$\text{Prob}(\mathbf{Y} \in \text{Sphere}(\mathbf{y}, d\mathbf{y})) = p(\mathbf{y}) d\mathbf{y}$$

Nonlinearities

- probabilistic descriptions not only for high dimensional systems
- but also for nonlinear, dissipative, lowdimensional systems (z.B. $\text{dim}(\mathbf{x}) = 3$), Eckman and Ruelle, 1985
- Lorenz (1963): butterfly effect (actually a sea gull effect)
- minimal differences in initial conditions magnify exponentially
- dissipation counteracts and keep the phase space trajectories on the attractor
- two initially indistinguishable states evolve into two randomly selected states on the attractor
- positive Lyapunov exponents
yet another interpretation

a system far from thermodynamical equilibrium

- differential solar insolation, high at the equator, low at polar latitudes (black body radiation at ∼ 6000 K)
- differential loss of radiative energy by thermal emissions at ∼ 250-280K, higher in low latitudes than at higher latitudes
- continuously externally driven system
- continuously generating entropy
- the climate system state probability $p(y)$ (climate-pdf) can not computed from first principles

- climatology now is the description of the climate-pdf and its dynamics (similar as in statistical physics of non-equilibrium systems)
- direct computation e.g. by a master equation or even only a Fokker-Planck equation prohibitive due to the high dimensional problem
- climate modelling can offer a first solution if the climate model is low dimensional ($< 20 \sim 100$)
- only reasonable solution: estimate properties of the climate pdf from data
- unfortunately reality provides only one realisation / one sample, single case probability, non-ergodic
- climate modelling can offer a second solution: Monte Carlo simulations
climate modelling

**climate system: a system of coupled cycles**

- energy cycle based on the First Law
- water cycle based on the conservation of water mass
- angular momentum cycle based on angular momentum conservation/Newton's law
- trace substances cycles based on mass conservation and law of mass action
- and inert mass conservation in each subsystem

Total energy equation in the atmosphere, valid on scales $L_x > 20 \text{ km}$, $T > 1 \text{ h}$ under hydrostatic assumption

\[
\frac{\partial}{\partial t} \left( c_p T + Lq + \frac{v_H^2}{2} \right) + \nabla H (\bar{v}_H (c_p T + Lq + \Phi + \frac{v_H^2}{2})) + \\
+ \frac{\partial}{\partial p} \left( \omega (c_p T + Lq + \Phi + \frac{v_H^2}{2}) \right) = \\
g \frac{\partial}{\partial p} \left( Q + \underbrace{H + LE}_{\text{turbulent and convective subscale fluxes}} \right) \tag{1}
\]

- turbulent: Kolmogorov-Prandtl like threedimensional turbulence
- convective: thunderstorm related turbulence
Total energy equation (atmosphere), vertical average

\[
\int \left( \frac{\partial}{\partial t} (c_p T + Lq + \frac{v_H^2}{2}) \right) \frac{dp}{g} + \int \left( \nabla \cdot (\vec{v}_H (c_p T + Lq + \Phi + \frac{v_H^2}{2})) \right) \frac{dp}{g} = (Q(p = 0) - (Q + H + LE)(p_b))
\]

if \( \omega(p_b) = 0 \) on the resolved scales (no atmospheric mass transport into/out off the surface of the solid Earth or ocean

The atmospheric water budget

- total water concentration \( q_T \) in the atmosphere
- sum of water vapour concentration \( q \), liquid water \( q_l \) and frozen water \( q_I \)
- mass weighted vertical average gives the total water substance in an atmospheric column \( m_T = \int \rho q_T dz \) with the budget equation

\[
\frac{d}{dt} m_T = \frac{\partial}{\partial t} m_T + \nabla \cdot (\vec{v}_h m_T) = E - P
\]

- \( m_T \) is called precipitable water expressed as height of a liquid water column
- \( E \) Evapo-Transpiration mass flux of water (MKS unit \( \text{kgm}^{-2}\text{sec}^{-1} \))
- \( P \) precipitation mass flux of water
Oceanic water budget
- ocean water is a solution of salt in freshwater
- salt concentration \( s \) and freshwater concentration \( 1 - s \)
- budget equation for the vertical integral of the salt concentration \( S = \int s \, dz \)

\[
\frac{d}{dt} \left( \int \rho_w (1 - s) \, dz \right) = -\rho_w \frac{d}{dt} S = P - E
\]

- water sources of the atmosphere are salt source for the ocean

Energy budget equation time averaged and vertically averaged in atmosphere and ocean

\[
\frac{\partial}{\partial t} e + \nabla_h \tilde{H}_e = \bar{Q}(p = 0)
\]

energy and water budget equation vertically averaged for the atmosphere

\[
\frac{\partial}{\partial t} e_A + \nabla_h \tilde{H}_{e,A} = \bar{Q}(p = 0) - (\bar{Q} + \bar{H} + L\bar{E})
\]

\[
\frac{\partial}{\partial t} m_T + \nabla_h \tilde{H}_m = \bar{E} - \bar{P}
\]
The Basics

The Energy and Water cycles

energy and water budget equation vertically averaged for the ocean

\[
\frac{\partial}{\partial t} e_O + \vec{\nabla} h \vec{H}_{e,O} = \vec{Q} + \vec{H} + L \vec{E}
\]

\[
\frac{\partial}{\partial t} S + \vec{\nabla} h \vec{H}_S = \vec{E} - \vec{P}
\]

boundary conditions along the coastlines

\[
\vec{H}_{e,O} \vec{n}_o = Q_{e,O} \sim 0
\]

\[
\vec{H}_S \vec{n}_o = Q_S = \sum_k Q_{S,K} \delta(\vec{r} - \vec{r}_k)
\]

prognostic modelling is not possible: the system of equations is not closed

- transports \( \vec{H} \) can not be calculated
- without considering the angular momentum budget in atmosphere and ocean
- mean \( \vec{v}_h \) and random fluctuations \( \vec{v}' \) can not be calculated

but an inverse calculation is possible

- take \( Q, H, E, P \) and \( \vec{H} \) from observations
- incomplete data
- spoiled by observational and sampling errors

under stationary conditions \( \frac{\partial}{\partial t} e = 0, \frac{\partial}{\partial t} m_T = 0 \) the budget equations can be reduced to

\[
\vec{\nabla} h \vec{H} = R
\]
if observations and its errors can be characterized by
  ▶ transports $\vec{H}_{\text{obs}}$ with white Gaussian errors $\epsilon_H$
  ▶ energy sinks/sources $R_{\text{obs}}$ with white Gaussian errors $\epsilon_R$

find $\vec{H}$ and $R$ such that

$$J = \frac{1}{2} \int \left( (\vec{H} - \vec{H}_{\text{obs}})^T (\Sigma_H)^{-1} (\vec{H} - \vec{H}_{\text{obs}}) + \frac{(R - R_{\text{obs}})^2}{\sigma_R^2} \right) d\Omega + \int (\lambda (\vec{\nabla} \vec{H} - R)) d\Omega \overset{\lambda}{=} \text{Minimum}$$

a variational problem with the Euler-Lagrange system of equations

$$\vec{\nabla} \vec{H} - R = 0$$
$$\Sigma_H^{-1} (\vec{H} - \vec{H}_o) - \vec{\nabla} \lambda = 0$$
$$\frac{1}{\sigma_R^2} (R - R_o) - \lambda = 0$$

leading to an elliptical problem in $\lambda$

$$\vec{\nabla} (\Sigma_H \vec{\nabla} \lambda) - \sigma_R^2 \lambda = (\vec{\nabla} \vec{H}_o - R_o)$$
Top-of-the-atmosphere radiation budget (solar)

Top-of-the-atmosphere radiation budget (IR)
Top-of-the-atmosphere radiation budget (Net)

atmospheric energy budget (Diploma thesis Chris Wosnitza, 2011)
**atmospheric water budget** (Diploma thesis Chris Wosnitza, 2011)

![Diagram of atmospheric water budget](image)

Climate sensitivity: global mean temperature

- how large is the change \( \Delta T \) of the global mean temperature if the CO\(_2\) concentration are doubled?
- doubling CO\(_2\) concentration will give a change in radiative forcing \( Q' \): \( \Delta T = \lambda_0 Q' \)
- feedbacks in the climate system will enhance the pure temperature change \( \Delta T = \lambda_0 (Q' + C\Delta T) \)
- \( \lambda_0 \sim 0.3 - 0.31K(W/m^2)^{-1} \)
- equilibrium with ocean

\[
\Delta T = \frac{\lambda_0}{1 - f} Q'
\]
Sensitivity of global mean water content "precipitable water":

\[
W = \int_{\lambda}^{\lambda_f} \left( \int_{\phi}^{\phi_f} \frac{q}{g} \rho \, dq \, dz \right) a^2 \sin \phi \, d\lambda,
\]

relative change

\[
\frac{1}{W} \frac{dW}{dQ} = \frac{1}{W} \frac{dW}{dT} \frac{dT}{dQ}
\]

climate sensitivity

(after Roe and Baker (2007), Science, 318, 629ff)
The Basics
The Energy and Water cycles

\[
\frac{1}{W} \frac{dW}{dT} = \frac{1}{W} \frac{dT}{dW} \int q \, dp \, g \\
= \frac{1}{W} \frac{dT}{dW} \int \left( r \frac{R_l e_s(T)}{R_w p} \right) \, dp \frac{g}{g} \\
\approx \frac{1}{W} \int \left( r \frac{R_l}{R_w p} \frac{L_e_s}{R_w T^2} \right) \, dp \frac{g}{g} \\
\sim \frac{L}{R_w T^2} \approx 0.065 - 0.07 \text{K}^{-1}
\]

famous Clausius-Clapeyron constraint

Andreas Hense
The global climate system and the water cycle

Not correct for global mean precipitation:

\[
\frac{d}{dt} W = E - P \sim 0
\]

recycling time of atmospheric water \( \sim 10 \) days

Instead consider atmospheric energy budget (equilibrium with ocean)

\[
Q'_c + \frac{1 - f}{\lambda_0} \Delta T = H' + LE' = LE'(1 + \beta)
\]

Andreas Hense
The global climate system and the water cycle
- $Q'_c$ loss of energy by infrared radiation due to an increased CO$_2$ concentration
- $\frac{1-f}{\lambda_0} \Delta T$ gain of energy by absorption of radiative energy $Q'$ from the surface
- $\beta$ Bowen ratio $\frac{H}{E}$
- water balance $E' \sim P'$

$$\frac{Q'_c}{L(1+\beta)} + \frac{1-f}{\lambda_0 L(1+\beta)} \Delta T = P'$$

Precipitation sensitivity: $\frac{1-f}{\lambda_0 L(1+\beta)} \sim 37 \text{mm}(\text{yrK})^{-1}$
or about 3.7 % per K if $P \sim 1000 \text{mm}(\text{yr})^{-1}$

(after Allen and Ingram (2002), Nature, 419, 224)
Lecture: Hydrological Models

Andreas Güntner

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

What is a hydrological model?

Climate input data
Time series of, e.g.,
- Precipitation
- Temperature
- Solar radiation
- Air humidity

Model equations
representing water fluxes and storage processes

Model output
Time series of, e.g.,
- Water storage
- River discharge
- Groundwater recharge

Model parameters
- describing, e.g., topography, vegetation, soil characteristics
- conceptual parameters
Storage variations in the continental water cycle

Several storage components:
- Snow and ice
- Groundwater
- Soil moisture
- Surface water storage

Continental water balance equation

\[ \Delta S = P - E - R \]

Hydrological models

Detailed physically-based models

1. Precipitation
2. Evaporation
3. Runoff
4. Water storage change

Detailed physically-based models:
- Penetranfall
- Bodenübersickerfall
- Infilttration in Mikro- und Makrozonen
- Oberflächenfall
- Zwischenfall (längere, oberflächenfall)
- Wasserabtrag in der Bodenmatrix (Mikrobrüche)
- Interzessionsabfluss / Mikroquerschnitt
- Tiefenfiltration oder Tiefenabfluss
- Grundwasserniveau
- Bodenerosion
- Pflanzenkonzipitation
Detailed physically-based models

For example:
Differential equation for unsaturated flow in a porous medium (Richards equation):

\[
\frac{\partial}{\partial t} \left( k(\theta) \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial y} \left( k_y(\theta) \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z(\theta) \frac{\partial w}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \theta}{\partial t} \right) = \frac{\partial \theta}{\partial t} - S
\]

Limitations of detailed physically-based models

Example: Soil water fluxes

Real-world infiltration pattern in a soil

Richards equation assumes a homogeneous porous medium

Physically-based model with macropores
Conceptual models

Linear storage (bucket approach)

\[ Q = k \cdot S \]

- \( Q \): Outflow (runoff)
- \( k \): Storage coefficient
- \( S \): Actual storage volume

\[ Q_t = Q_0 \exp(-t / k) \]

Example: Soil and ground water fluxes

Linear storage approach used in many large-scale models

- \( Q \): Outflow (runoff)
- \( k \): Storage coefficient
- \( S \): Actual storage volume

\( k \) may be estimated/calibrated from observed discharge time series
Conceptual models

Example: Soil and ground water fluxes

Limitations:
- Model cannot be transferred to other areas
- Model can usually only be applied to situations for which it has been calibrated (poor for extremes, inter-annual variations, trends)

Conceptual models

Runoff generation by a non-linear response function at the 0.5° scale

\[ Q = P_{\text{eff}} \cdot \left( \frac{S}{S_{\text{max}}} \right)^b \]

- \( Q \): Runoff
- \( P_{\text{eff}} \): Effective precipitation
- \( S \): Actual soil water content
- \( S_{\text{max}} \): Maximum soil water storage
- \( b \): Calibration parameter

This equation is used in the WaterGAP global hydrology model (WGHM), for example.
Water storage – spatial variability

Tarrawarra catchment (Victoria, Australia) (Western & Grayson, 2001)

Wet conditions in winter
Dry conditions in summer

Tarrawarra catchment (Victoria, Australia) (Western & Grayson, 2001)
Representing spatial variability

For example:
Differential equation for unsaturated flow in a porous medium (Richards equation):

\[ \frac{\partial}{\partial x} \left( k_u(\theta) \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_u(\theta) \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_u(\theta) \frac{\partial \psi}{\partial z} + 1 \right) = \frac{\partial \theta}{\partial t} - S \]

Limitation: Very demanding in data / parameters

Variations of parameter values within a grid cell

Mosaic approach

Distribution function

(Beispiel: VIC-Modell)
Large-scale models of continental hydrology

**Land Surface Models**

- Land surface description in climate models
- Water balance
- Energy balance
- (Carbon fluxes)
- Vertical water fluxes, several soil layers
- High temporal resolution (minutes-hours)

**Global hydrological models**

- Water balance for grid cells / river basins
- Lateral water fluxes
- Routing in river network
- (Water use / consumption)
- Daily – monthly temporal resolution
- Conceptual process representation

---

**Surface runoff**

**Infiltration**

**Groundwater recharge**

**River discharge**

**Hydrograph**

**Subsurface runoff**

**Evapotranspiration**
Large-scale models of continental hydrology

WaterMIP (Water Model Intercomparison Project)

Haddeland et al. 2011, J. Hydrometeor.
Harding et al. 2011, J. Hydrometeor.
Large-scale models of continental hydrology

WaterMIP (Water Model Intercomparison Project)

Global simulation results (mean annual values 1985-1999)

LSM: Land Surface Model
GHM: Global Hydrological Model


Structure of large-scale hydrological models

Total continental water storage \( \approx \) GRACE

Processes and storages in brackets are not represented in all models

Harding et al. 2011, J. Hydrometeor.
Large-scale models of continental hydrology

1) WaterGAP Global Hydrology model (WGHM)

Total continental water storage change $\Delta S$:

$$\Delta S = \Delta S_{\text{canopy}} + \Delta S_{\text{snow}} + \Delta S_{\text{soil}} + \Delta S_{\text{groundwater}} + \Delta S_{\text{rivers}} + \Delta S_{\text{lakes/reservoirs}} + \Delta S_{\text{wetlands}}$$

2) Land Dynamics (LaD) World

$$\Delta S = \Delta S_{\text{snow}} + \Delta S_{\text{soil}} + \Delta S_{\text{groundwater}}$$

3) Global Land Data Assimilation System (GLDAS)

$$\Delta S = \Delta S_{\text{canopy}} + \Delta S_{\text{snow}} + \Delta S_{\text{soil}}$$

Variations of continental water storage

RMS variability of monthly values around annual mean for 2004 (in mm w.eq.)
Mean monthly water storage in different storage compartments (WGHM simulation results)

Variations of continental water storage

Köppen climate zones
Variations of continental water storage

Month of maximum water storage (based on WGHM simulation)

Günther et al. 2007, WRR

Water storage variations at the river basin scale

Basin-average water storage – GRACE filter effects

EBF: Effective basin function
(here: simple truncation at Lmax=60)

Longuevergne et al. 2010, WRR
Water storage variations at the river basin scale

**Basin-average water storage - GRACE filter effects**

**Effective basin function**

![Diagram of basin-average water storage - GRACE filter effects]

**Water storage variations at the river basin scale**

<table>
<thead>
<tr>
<th>Filter type</th>
<th>Bias of seasonal amplitude (mm)</th>
<th>Bias of seasonal phase (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: Gaussian filter (500km)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II: Swenson &amp; Wahr (2002) maximum allowed satellite error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III: Swenson &amp; Wahr (2002) min. satellite and leakage error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV: Seo et al. (2006) dynamic filter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V: Swenson &amp; Wahr (2006) decorrelation filter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>VI: Kusche (2007) decorrelation filter</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| River | Filter type | Bias of seasonal amplitude | Bias of seasonal phase | | |
|-------|-------------|---------------------------|-----------------------| |
| Amazon | I | -18 | -2 | 6 | 18 | -47 | 1 | 2 | 0 | 0 | -1 | -2 | -9 | |
| Amur | II | -20 | 7 | 8 | 40 | -64 | 2 | 5 | 1 | 4 | 2 | 9 | 3 | |
| Danube | III | -84 | -41 | -67 | -91 | -117 | -26 | 1 | 1 | 6 | -9 | -5 | 3 | |
| Ganges | IV | 5 | -1 | -1 | 4 | 1 | 1 | 1 | |
| Indus | V | -74 | 4 | 1 | 1 | |
| Lena | VI | 5 | -1 | -3 | 4 | 1 | 1 | |
| Longuevergne et al. 2010, WRR | | | | | | | | | | | | | |
Water storage variations at the river basin scale

### GRACE filter-induced bias in basin-average water storage

<table>
<thead>
<tr>
<th>Filter type</th>
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<th>III</th>
<th>IV</th>
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### Consistent filtering of GRACE and hydrological model data needed before comparing or merging


---

Flow chart for estimating basin-scale water storage variations from GRACE level 2 spherical harmonic products

Longuevergne et al. 2010, WRR
What is a hydrological model?

Climate input data
- Precipitation
- Temperature
- Solar radiation
- Air humidity

Model equations
- Representing water fluxes and storage processes

Model output
- Time series of, e.g.,
  - Water storage
  - River discharge
  - Groundwater recharge

Model parameters
- Describing, e.g., topography, vegetation, soil characteristics
- Conceptual parameters

Large-scale models of continental hydrology: parameters

Vegetation parameters (partly time-variable)
- Leaf area index
- Albedo
- Interception storage capacity
- Stomata resistance
- Aerodynamic roughness
- Canopy height
- Root depth
- ...

Soil parameters
- Porosity
- Field capacity
- Hydraulic conductivity
- Soil depth
- Capillary head as function of water content
- Heat transport and storage
- ...

Snow parameters, e.g., density, water and energy storage and transport parameters

Other hydrological / hydraulic parameters
- Slope gradient
- River cross section geometry
- Lake/reservoir storage capacity
Conceptual models

Example: Soil and ground water fluxes

Linear storage approach used in many large-scale models

\[ Q = k \cdot S \]

- \( Q \): Outflow (runoff)
- \( k \): Storage coefficient
- \( S \): Actual storage volume

\( k \) may be estimated/calibrated from observed discharge time series

What is a hydrological model?

Climate input data
Time series of, e.g.,
- Precipitation
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- Solar radiation
- Air humidity

Model equations
representing water fluxes and storage processes

Model output
Time series of, e.g.,
- Water storage
- River discharge
- Groundwater recharge

Model parameters
- Describing, e.g., topography, vegetation, soil characteristics
- Conceptual parameters

Model calibration

Observed data
usually river discharge
Water balance of a river basin:

\[ P = E + Q + \Delta S \]

- **P**: Precipitation
- **E**: Evapotranspiration
- **Q**: Runoff
- **\( \Delta S \)**: Water storage change

**Model input**
- Simulated in the model based on meteorological input data

**Traditional calibration variable**
- Simulated basin-average soil moisture for the two model versions

*(Data from Merz & Blöschl, TU Wien)*
Water balance of a river basin:

\[ P = E + Q + \Delta S \]

**P**: Precipitation  
**E**: Evapotranspiration  
**Q**: Runoff (measured time series of river discharge)  
**\( \Delta S \)**: Water storage change (basin-average values from GRACE)

Multi-criterial calibration of hydrological models

Example: Amazon basin

**Calibration run 1**

**Calibration run 2**

- Observation
- Original model
- Calibrated model
Multi-criterial calibration of hydrological models

**Climate input data**
- Time series of, e.g.,
  - Precipitation
  - Temperature
  - Solar radiation
  - Air humidity

**Model equations**
- Representing water fluxes and storage processes

**Model parameters**
- Describing, e.g.,
  - Topography
  - Vegetation
  - Soil characteristics
  - Conceptual parameters

**Multi-criterial calibration**

**Observed data**
- River discharge
- GRACE water storage
- Soil moisture
- Water level (altimetry)
- ET products
- …

Integration of GRACE data into large-scale hydrological models

Water storage variations from GRACE are not routinely incorporated into hydrological models so far.

Only few examples:

- Zaitchik et al. (2008): Data assimilation, Kalman filter (Mississippi, CLSM)
- Werth et al. (2009, 2010): Multi-objective calibration (large basins worldwide, WGHM)
- Lo et al. (2010): Multi-objective calibration (Illinois, CLM)
- Milzow et al. (2011): Multi-objective calibration (Okavango, SWAT)
Availability of model output

Simulation results of global hydrological models usually are available on request from the modellers only.

Only few free download sites:

- GLDAS (Mosaic, CLM, Noah, VIC)
  http://disc.sci.gsfc.nasa.gov/services/grads-gds/gldas

Lessons learned

- Large uncertainties / differences in simulation results of global hydrological models
  → try to use multi-model ensembles

- Preserve consistency when comparing / combining GRACE data and hydrological model data:
  → GRACE sees total water storage variations – a model should represent all storage compartments
  → do the same signal processing (filtering) for GRACE and model data
  → or: correct for filter effects (bias, leakage) in final GRACE-based basin water storage variations

- Comprehensive improvement of models by using various observation data (satellite-based) as multiple constraints
Lecture: Surface loading

Volker Klemann, GFZ

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoß

Motto

• Interaction between surface processes and solid earth
Overview

1. Mathematical prerequisites
   - Surface load – loading processes – separation of signals – solid earth response – Representation by spherical harmonics– spheroidal/toroidal motion

2. The sea-level equation (SLE)

3. Reference systems
   - Position and orientation in space

4. Glacial-isostatic adjustment
   - The process – field equations – solution

1. Mathematical prerequisites

- Concept of a surface load
- Processes acting as a surface load
- Separation of signals
- Solid earth response
  - Excursion to physical meaning of Legendre degrees 0 and 1
  - Toroidal motion
1. Concept of surface load

Surface mass density at earth surface, $a$, is defined as

$$\sigma(a, \Omega) := \int_{r_0} \rho_\sigma(r, \Omega) dr .$$

$\Omega = (\theta, \phi)$ is the coordinate pair,

$r_0$ is the considered radial range,

$\rho_\sigma$ is the density distribution of the load.

1.2 Processes acting as a surface load

- Hydrological water storage
- Mass redistribution inside the ocean
- Glacial melting
- (Atmosphere)
1.3 Separation of signals

- Spatial distribution
  - Masking
  - Spectral analysis
  - EOF ?

- Temporal behavior
  - Spectral analysis (annual signal)
  - Separation of linear trend
  - EOF ?

Temporal variations of geoid from GRACE
courtesy of I. Sasgen
1.4 Solid earth response

- Representation of fields
- Linear response
- Relation between load and displacements expressed by load-Love numbers

Spectral representation of field quantities

Displacement
\[ u(r, \Omega) = \sum_{lm} \left[ U_{lm}(r) S_{lm}^{-1}(\Omega) + V_{lm}(r) S_{lm}^{+1}(\Omega) + W_{lm}(r) S_{lm}^{(0)}(\Omega) \right] \]

Potential perturbation
\[ \varphi_1(r, \Omega) = \sum_{lm} \Phi_{lm}(r) \gamma_{lm}(\Omega) \]
1.4.2 Toroidal motion

\[ u_{\text{toro}}(r, \Omega) = \sum W_{jm}(r) S_{jm}^{(0)}(\Omega) \]

where

\[ S_{jm}^{(0)} = (e_r \times \nabla_{\Omega}) Y_{jm} \]

Motion is not excited as long as

1. the earth structure is spherical symmetric
2. the loading is only acting in vertical direction

Convolution integral

Surface mass density

\[ \sigma(\Omega) = \sum \Sigma_{jm} Y_{jm}(\Omega) . \]

Spherical symmetry of Earth structure

\[ \Rightarrow \varphi(\Omega) = a^2 \int_{\Omega_0} g_\varphi(\gamma) \sigma(\Omega') \, d\Omega' , \quad \gamma = |\Omega - \Omega'|, \]

with Green's function,

\[ g_\varphi(\gamma) = \sum_l G_l^\varphi P_l(\cos \gamma) , \]

\[ \Rightarrow \varphi(\Omega) = 4\pi a^2 \sum \Sigma_{jm} Y_{jm}(\Omega) . \]
Green’s functions – Load Love numbers

Displacement of reference potential and surface:

\[ g_e(y) := \frac{a}{M_e} \sum_l (1 + k_l) P_l(\cos y), \]
\[ g_u(y) := \frac{a}{M_e} \sum_l h_l P_l(\cos y) \]

where \( h_l \) and \( k_l \) are the load Love numbers.

\[ e(\Omega) = \frac{3}{\rho} \sum \frac{1 + k_l}{2l + 1} \sum_m Y_{lm}(\Omega) \]
\[ u(\Omega) = \frac{3}{\rho} \sum \frac{h_l}{2l + 1} \sum_m Y_{lm}(\Omega) \]

Characteristics of Love numbers

- **Response of solid earth**
  - Spherical symmetric (average crust, no difference between continent and ocean)
  - Elastic, compressible
  - The effect of self gravitation is considered
    
    see GIA section

- **h, k, I describe vertical, potential and horizontal displacement.**

- **2 process**
  - load Love numbers (response to surface pressure)
  - tidal love numbers (response to tidal forcing)

- **They are valid for instantaneous processes**

- **Anelasticity is not considered**
Load Love numbers

1.4.1 Legendre degrees 0 and 1

\[ \int_{\Omega} Y_m \, d\Omega = \frac{\sqrt{4\pi}}{4\pi} \delta_{0m}, \]

\[ \int_{\Omega} \sigma \, d\Omega = 0 \quad \Rightarrow \quad [U, E]_{00} = 0 \]

\[ \int_{\Omega} S_{jm}^{(A)} \, d\Omega = \sqrt{\frac{4\pi}{3}} \delta_{1j} (2 \delta_{A1} + \delta_{A-1}) e_m \]

\[ u_{\text{CF}} := \frac{1}{A} \int_{\partial V} u \, dS \propto [U, V]_{1m} \Sigma_{1m} \]

\[ u_{\text{CM}} := \frac{1}{M} \left( \int_{V} \rho u \, dV + \int_{\partial V} \sigma r \, dS \right) \propto \Phi_{1m} \Sigma_{1m} \]
How is the water distributed in the loaded bathtub if the ice is melting?

\[ m_{\text{load}}(\Omega) = m_{\text{ice}}(\Omega) + m_{\text{oce}}(\Omega) = 0. \]

1. Definition of geoid
   Physicist – Geodesist
2. Definition of sea level variation
   Static equilibrium surface – deforming earth
3. Ocean function
   Where can water be filled into?
4. Moving coastlines
   A bathtub with slopes
5. Coupling
   Loading effect of redistributed water
6. Formulation of SLE
   Integral equation
7. Features of its solution
2.7 Features of solution

- Ice load
- Conservation of mass
- Gravitational attraction
- Deformation
- Change of geoid
- Melting of ice

2.1 The concept of geoid

The geoid is that equipotential surface which would coincide exactly with the mean ocean surface of the Earth, if the oceans were in equilibrium, at rest (relative to the rotating Earth), and extended through the continents (such as with very narrow canals).

According to C.F. Gauss, who first described it, it is the “mathematical figure of the Earth”, a smooth but highly irregular surface that corresponds not to the actual surface of the Earth’s crust, but to a surface which can only be known through extensive gravitational measurements and calculations.

Wikipedia
Geodesists view?

The reference geoid is defined by a specific potential value (IERS-2010 convention)

\[ W_0 = 62636856.0 \, \text{m}^2 \, \text{s}^{-2} \pm 0.5 \, \text{m}^2 \, \text{s}^{-2} \]

So, the geoid corresponds to the equipotential surface, \( W_0 \)?

\[ \Rightarrow e(\Omega) = \varphi_1(\Omega)/g_0 \]

We call this quantity potential displacement.

2.2 Definitions of sea level

The geoid is defined as

\[ n(\Omega, t) := e(\Omega, t) + h_{\text{wl}}(t) \]

with \( h_{\text{wl}} \) the distance between the reference-potential height and the potential height which the current sea level is following.

Then, relative sea level:

\[ h_{\text{RSL}}(\Omega, t) := [n - u](\Omega, t) - [n - u](\Omega, t_0), \]

altimetric sea level:

\[ h_{\text{alt}}(\Omega, t) := n(\Omega, t) - n(\Omega, t_0) \]
Definitions of datum

- Surface displacement, $h_{\text{surf}}$, is measured by GPS and measured in a specific reference system.
- Altimetric sea level, $h_{\text{alt}}$, is measured in a specific reference system.
- Relative sea level: Sea level change at a tide gauge
- The relative sea level is the difference, $h_{\text{alt}} - h_{\text{surf}}$, only if the reference systems, $+$, are defined in the same way.

2.3 The ocean function

$$O(\Omega, t) := \begin{cases} 0 & \text{if } T(\Omega, t) > 0 \\ 1 & \text{if } T(\Omega, t) \leq 0 \end{cases}$$

with time-dependent topography

$$T(\Omega, t) = T_0 - h_{\text{RSL}}(\Omega, t)$$
2.4 Moving coastlines

During last glacial maximum large continental-shelf areas were above sea level.

2.6 The sea-level equation of Farrell and Clark (1976)

We assume that all perturbations vanish at $t = 0$.

$$h_{rfl}(\Omega, t) = [h_{wl}(t) + e(\Omega, t) - u(\Omega, t)]O(\Omega, t),$$

with

$$h_{wl}(t) = \frac{-m_{ice}(t)}{\rho_{oce}A_{oce}(t)} - \frac{1}{A_{oce}(t)} \int_{\Omega} [e(\Omega, t) - u(\Omega, t)]O(\Omega, t) d\Omega$$

and

$$A_{oce}(t) = \int_{\Omega} O(\Omega, t) d\Omega$$
Iterative procedure

At each time, \( t \):

\[
(e - u)_i(\omega) = g_{e-u} \ast [m_{i\text{ce}}(\Omega) + h_{i-1}^{\text{rel}}(\omega) \rho_w O(\omega)],
\]

\[
h_i^{\text{rel}}(\omega) = h_i^{\text{rel}} + (e - u)_i(\omega) O(\omega),
\]

\[
h_i^{\text{rel}} = - \int_{\Omega} m_{i\text{ce}}(\omega) \frac{1}{\rho_w A_o} \int (e - u)_i(\omega) O(\omega) d\omega,
\]

\[
h_0^{\text{rel}} = - \int m_{i\text{ce}}(\omega) \frac{1}{\rho_w A_o}.
\]

3. Reference systems

- **Reference frame**
  
  A set of axes within which to measure the position, orientation, and other properties of an object or a process

- **Geodetic data (more than one datum)**

  are used in geodesy to translate positions indicated on their products to their real position on earth

- 3 coordinates for orientation
- 3 coordinates for position
3.1 Orientation

1. no surface net rotation (NF) \( \int_{\partial B} \mathbf{e}_r \times u dS = 0 \)
2. conservation of total momentum (NM) \( \int_B \rho \mathbf{e}_r \times u dV + \int_{\partial B} \mathbf{e}_r \times \sigma dS = 0 \)
3. no lithosphere net rotation (NL) \( \int_{B_L} \rho \mathbf{e}_r \times u dV = 0 \)
4. no internal rotation (NE) \( \int_B \rho \mathbf{e}_r \times u dV = 0 \)
5. no mantle rotation (NMa) \( \int_{B_M} \rho \mathbf{e}_r \times u dV = 0 \)

3.2 Position

1. centre of mass (CM) \( \int_B \rho r dV + \int_{\partial B} \sigma r dS = 0 \)
2. centre of figure (CF) \( \int_{\partial B} u dS = 0 \)
3. centre of deformation (CD) \( \int_B u dV = 0 \)
4. centre of internal masses (CE) \( \int_B \rho r dV = 0 \)
CM and CF motion due to GIA

- Surface loading
  - CM towards load
  - GC in opposite direction
- Viscoelastic compensation
  - Downward displacement
  - CM away from load
  - CF away from load
- After deglaciation
  - CM first away from load area than moves towards load centre
  - CF towards load area

GIA contribution to present-day processes

- Prediction of GIA component for present-day dynamics and its environmental impact

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RSL change from SLIs

- Land uplift – representation by fossil samples (SLI) deposited subsequently

![Graph showing age of sample vs. RSL height](image)

4. Glacial Isostatic Adjustment (GIA)

- GIA describes the kinematic and dynamic response of the earth’s interior to surface loading related to the glacial periods
  - Reconfiguration inside the solid earth
  - Coupling to ice/ocean mass redistribution
- Mathematical formulation
  - Continuum mechanical field equations describing
    - the momentum of a visco-elastic material filling a planet
    - the gravitational perturbation due to reconfiguration
    - Excitation by surface and body forces
Features of GIA modelling

- PREM structure for shear modulus and density
- Viscosities: \( \eta_{UM} = 0.6 \times 10^{20} \text{ Pa s} \), \( \eta_{LM} = 1 \times 10^{22} \text{ Pa s} \)
- Elastic lithosphere of variable thickness
  Predefined ice history (e.g. ICE5G)

- S-FE formulation (Martinec, GJI, 2000)
  - incompressible
  - Compressibility Tanaka et al. (2011)
  - self-gravitating
  - rotating
  - hydrostatically pre-stressed

- Uniqueness conditions
  - centre of mass
  - no surface net-rotation

Main aspects of process

- Loading force = flexure of lithosphere + buoyancy
- Elasticity
- gravity
- Fluid dynamics
Flow field at surface

1D earth structure
- standard viscosity model with
  $\eta_{UM} = 6 \times 10^{20}$ Pa s
  $\eta_{LM} = 1 \times 10^{22}$ Pa s
- ICE3G history
- fixed coastlines

Flow field in earth’s mantle

Mid Ocean Ridge (MOR)
Ice sheets:
Laurentide (LIS)
Greenland (GIS)
Fennoscandia (FIS)
Last glacial cycle

- Glaciated regions on northern hemisphere:
  - North America, Greenland
  - Fennoscandia
- on southern hemisphere
  - Antarctica

2.7 Features of solution

- Ice load
- Conservation of mass
- Gravitational attraction
- Deformation
- Change of geoid
- Melting of ice
- Conservation of mass + change of geoid
- Uplift
4.1 Field equations describing the solid earth response

Lagrange' formulation of equation of motion – Potential equation – Constitutive equation – Continuity equation

\[ \nabla \cdot \tau - \rho_0 \nabla \varphi_1 + \nabla \cdot (\rho_0 u) \varphi_0 - \nabla (\rho_0 u \cdot \nabla \varphi_0) = 0 \]

\[ \nabla^2 \varphi_1 + 4\pi G \nabla \cdot (\rho_0 u) = 0 \]

\[ \dot{t} = \dot{t}^E - \frac{\mu}{\eta} (\dot{t} - \nabla I), \quad \dot{t}^E = \nabla I + \mu (\nabla u + \nabla u^T) \]

\[ \nabla \cdot u = 0 \]

inside the earth, \( B \): Displacement, \( u \) – Stress, \( \tau \) – Pressure, \( I \) – Potential, \( \varphi_1 \). — Material parameters: Density, \( \rho_0 \) – Shear modulus, \( \mu \) – Viscosity, \( \eta \)
Boundary conditions

Loading of surface mass – free slip – continuity of gravitational potential – potential of surface mass

\[ e_r \cdot \tau^+ \cdot e_r = -g_0(a) \sigma \]
\[ \tau^- \cdot e_r - (e_r \cdot \tau^- \cdot e_r) e_r = 0 \]
\[ [\varphi_1]^+ = 0 \]
\[ [\nabla \varphi_1]^+ \cdot e_r + 4\pi G \rho^- (u^- \cdot e_r) = 4\pi G \sigma \]

Green’s function for the viscoelastic case

Convolution in space and time

\[ \varphi(\Omega, t) = \int_{-\infty}^{t} \int_{\Omega_0} g_{\varphi}(y, t - t') \sigma(\Omega', t') \, d\Omega' \, dt' \]

Features of \( g_{\varphi} \)

• fading memory
• objectivity
• ...
Maxwell rheology

- Linear transition from elastic to viscous material behavior

\[ \dot{\tau} + \frac{\mu_0}{\eta} \tau = 2 \mu \varepsilon^d \]

\[ \tau(s) = \mu(s) \varepsilon^d(s) \]

\[ \mu(s) = \frac{2 \mu_0 s}{s + \mu_0/\eta} \]

Normal mode approach

  - Representation of fields in spatial domain by series expanded in spherical harmonics for lateral directions
  - Replacement of relaxation function by complex shear modulus (in contrast to the free oscillation problem, a Laplace transformation is applied.)
  - Representation of solution by set of eigen modes, which are determined by the roots of the secular determinant (Inverse Laplace transformation - Bromwich path – Residue theorem)
Solution of radial part

\[ \frac{\partial}{\partial r} Y = A Y \]

The formal solution is \[ Y(r) = e^{Ar} C \]

Being \( Y \) the solution at \( r \):

\[ Y(r) = L(r) C \]

with \( L \) the fundamental matrix of \( A \).

---

Transformation to time domain

\[ \mu(s) = \frac{\mu_0}{1 + (s \tau)^{-1}} \]

\( s \)-dependent solution has to be transformed into spatial domain.

\[ f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f(s) e^{st} ds \]

Closing Bromwich path, we can replace the integral by the residues of \( f(s) \).

\[ f(t) = \sum_{j=1}^{M} \text{Res} \{ f(s) e^{st} \} |_{s=s_j} \]

\[ \text{Res} \left\{ \frac{p(s_j)}{q(s_j)} \right\} = \frac{p(s_j)}{d/ds q(s_j)} |_{s=s_j} \]

\[ s(a, t) = K(t) = K_0 \delta(t) + \sum_{j=1}^{M} K_j(a) e^{\epsilon_j t} \]
Meaning of relaxation modes

Geocenter motion

\[ u_{gc} := u_{cf} - u_{cm} \]

or in components

\[ u_{g_c}^x = -\frac{1}{2}\sqrt{\frac{2}{\pi}} \text{Re}\{ U_{11} + 2V_{11} + 3F_{11}/g_0 \} \]

\[ u_{g_c}^y = \frac{1}{2}\sqrt{\frac{2}{\pi}} \text{Im}\{ U_{11} + 2V_{11} + 3F_{11}/g_0 \} \]

\[ u_{g_c}^z = \frac{1}{2}\sqrt{\frac{1}{\pi}} (U_{10} + 2V_{10} + 3F_{10}/g_0) \]
Evolution of motions in the CE realization during last glacial cycle

- CF is delayed and opposite to CM
- Amplitude of GC is largest during LGM and reaches 70 m
- After deglaciation CM is negligible and GC is dominated by the delayed CF

Influence of lower-mantle viscosity on GC motion

- Variation in direction of GC motion ~ 2000 km
- Velocity of motion varies by almost one magnitude
- Largest sensitivity between $10^{21}$ and $10^{23}$ Pa s
4.2 Solution of field equations

- Advanced modelling with spectral—finite elements
  - Solution directly in the time domain
  - Lateral variations in viscoelastic parameters are possible
  - Dynamic coupling to other processes is more easy to establish
  - Consideration of non-linear viscoelasticity
**Elastic thickness of Lithosphere**

- Depth defined by characteristic isotherm (1100 °C)
- Mosaic
  - Continental lithosphere from thermal data (Artemieva, Tectonophysics, 2003)
  - Oceanic lithosphere from ocean floor ages (Müller et al., JGR, 1997)
  - Plate boundaries from Bird (G3, 2003)

---

**Induced horizontal motions**

(Klemann, et al, 2008, JGdyn)
Divergence and vorticity

Klemann, et al., 2008, JGdyn

Degree variances of surface motion

Influence of plates

- Equipartitioning of spheroidal and toroidal component of GIA induced horizontal motions for $j > 3$

- Equipartitioning appears in plate motions driven by convective flow (e.g. Čadek & Ricard, EPSL, 1992)

- Toroidal motion vanishes for $j = 1$ due to uniqueness condition of no surface net-rotation

Klemann et al. 2008, J. Geodyn.
Solid earth – ice coupling

- Polythermal ice model
  - forced by atmosphere
    - air temperature
    - precipitation
  - forced by ocean
    - sea level
    - grounding line
  - forced by solid earth
    - heat flow
  - response to loading
    - viscoelastic earth

- Dynamic coupling
  - $h_{\text{ICE}} = t_{\text{ICE}} \cdot u$
  - flow inside ice sheet

Summary

- Love numbers
  - Elastic response of the earth’s interior to surface processes
  - Spherical symmetry
  - Gravitational consistent

- Sea level equation
  - Mandatory to describe static redistribution of water inside the ocean for a loading process

- Glacial isostatic adjustment
  - Most important secular process for geodetic observables
  - Earth interior is usually represented by a simplified structure
Practical: Ice and loading

Volker Klemann, Andreas Groh

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

Main tasks

- Convolution
- Analysis
- Synthesis
Solution of sea-level equation (sle_pt.m)

1. Definition of global parameters.
2. Definition of Input files. The LLN, the topography and the ice-height changes are provided.
3. Read in of LLN and definition of the Green’s function needed for the SLE
4. Determination of ocean function: read in of topography – define 1/0 grid
5. Initialise iteration procedure: initialise working grid – determine ocean surface – read in of load and transfer to surface-mass density – synthesise – initialise sea level
6. Iterate sea-level equation: convolution – determine and add new equivalent sea level

6. Iterative procedure

At each time, $t$:

$$(e - u)_i(\Omega) = g_{e-u} \ast [m_{\text{ice}}(\Omega) + h_{i-1}^{\text{sle}}(\Omega) \rho_w O(\Omega)],$$

$$h_i^{\text{sle}}(\Omega) = h_i^{\text{wl}} + (e - u)_i(\Omega) O(\Omega),$$

$$h_i^{\text{wl}} = - \int_\Omega m_{\text{ice}}(\Omega) \frac{1}{\rho_w A_o} - \frac{1}{A_o} \int (e - u)_i(\Omega) O(\Omega) d\Omega,$$

$$h_0^{\text{sle}} = - \int m_{\text{ice}}(\Omega) \frac{1}{\rho_w A_o}.$$
First exercise

Your task:

- Program the convolution
- Run the code
- Discuss the convergence of the solution
- **Keep the solution for the second exercise**

Analysis of present day fields (**compare_pt.m**):

- 1. i/o
- 2. i/o
- 3. i/o
- 4. Determination of ocean function: read in of topography – define 1/0 grid
- 5. Read in of GRACE stokes trend coefficients and conversion into geoid trend coefficients
- 6. Read in of GIA equipotential surface trend coefficients
- 7. Read in of the relative sea-level change from the last iteration of Exercise 1 and the ice
- height changes
- 8. Combination of the later and transformation into spherical harmonic coefficients
- 9. Perform the convolution in order to derive present-day geoid changes
- 10. Combination of GIA and present-day geoid changes and comparison to GRACE results
• On the next slides the solutions of the practicals are presented.
Lecture: Ocean Dynamics

Martin Losch, Henryk Dobslaw

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

Introduction:
Sea level variability at Bermuda
- tide gauge at Biological Station (8 years)
- surface pressure
- local wind speed
- air temperature
- hydrographic data (bi-weekly)
- secondary tide gauge (~3 km away)

**TABLE 4. Sea-Level Variance**

<table>
<thead>
<tr>
<th>Major Source</th>
<th>Percentage of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ms tide</td>
<td>60</td>
</tr>
<tr>
<td>All linear tides</td>
<td>70</td>
</tr>
<tr>
<td>Radiational tides</td>
<td>0.5</td>
</tr>
<tr>
<td>Overides (including thermal tides $S_0, S_2$)</td>
<td>negligible</td>
</tr>
<tr>
<td>Atmospheric pressure</td>
<td>8</td>
</tr>
<tr>
<td>Atmospheric pressure and winds</td>
<td>14</td>
</tr>
<tr>
<td>Dynamic height variations to 1500 db</td>
<td>3</td>
</tr>
<tr>
<td>Least count noise</td>
<td>0.2</td>
</tr>
<tr>
<td>Total</td>
<td>87</td>
</tr>
</tbody>
</table>

Wunsch (1972)
Bermuda Sea Level (Carl Wunsch, 1972)

- tide gauge at Biological Station (8 years)
- surface pressure
- local wind speed
- air temperature
- hydrographic data (bi-weekly)
- secondary tide gauge (~3 km away)

<table>
<thead>
<tr>
<th>Major Source</th>
<th>Percentage of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>M2 tide</td>
<td>60</td>
</tr>
<tr>
<td>All lunar tides</td>
<td>70</td>
</tr>
<tr>
<td>Radiational tides</td>
<td>0.2</td>
</tr>
</tbody>
</table>
| Overides (including thermal tides S
  4, S6)                              | negligible          |
| Atmospheric pressure                | 8                   |
| Atmospheric pressure and winds      | 14                  |
| Dynamic height variations to 1000 db| 3                   |
| Least count noise                   | 0.2                 |
| Total                               | 87                  |

Wunsch (1972)

Objective of Oceanography

- Describe three-dimensional ocean circulation and associated transports of energy (heat) and matter (salt, nutrients, etc.)
- Methods are in principle:
  - observations
  - mathematical and numerical modelling
  - combination of observations and models
1. Properties of Sea Water
2. Ocean Dynamics: equations of motion, common approximations and consequences
3. Non-Tidal short-term variability: effects of pressure, effects of wind, recent models, consideration for GRACE
4. Ocean Tides: tidal potential, Laplace’s Tidal Equations, classes of solutions to LTE, recent models, semi-empirical models from GRACE & altimetry
5. De-aliasing
Properties of Sea Water

Constituents of sea water: fresh water, minerals/salt (order 35 g/kg) observable properties:
• temperature: (degree celsius) in-situ (-> potential and conservative temperature)
• salinity: (practical salinity scale), do not use “psu” (practical salinity unit) as “unit”
• density: function of temperature, salinity, pressure (depth), so-called equation of state (EOS). Most recent official EOS is TEOS10 (McDougall et al. 2010).
  • flow is driven by horizontal density gradients
  • flow it predominantly along isopycnals -> potential density, neutral density

• sea water is compressible, water column can expand and contract; bottom pressure changes through mass changes (mass flux at surface, lateral mass flux), but not through expansion/contraction due to warming/cooling

Properties of Sea Water: Temperature

• in-situ temperature
• potential temperature: (temperature of a water parcel after being lifted adiabatically to a reference level, usually the surface)
• conservative temperature

• TEOS10 (McDougall et al 2010) uses concept of conservative temperature (potential enthalpy)

ICDC, WOCE data
Properties of Sea Water: Salinity

- practical salinity scale (pss), not a “unit”
- absolute salinity (SA): g of dissolved minerals per kg sea water

ICDC, WOCE data

Properties of Sea Water: Density = R(S,T,p)

In-situ density, potential density (density of a water parcel after lifting adiabatically to a reference level (e.g. surface)), neutral density. Convention: \( \sigma = \rho - 1000 \text{ kg/m}^3 \)

Data source: Levitus (1994)
Properties of Sea Water: Density

### Table 2.1
Values of density in situ for fresh and sea water (kg m\(^{-3}\))

<table>
<thead>
<tr>
<th>Sea pressure (kPa)</th>
<th>Approx. depth (m)</th>
<th>Frozen water</th>
<th>0°C</th>
<th>30°C</th>
<th>Salinity</th>
<th>0°C</th>
<th>30°C</th>
<th>18°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>993.8</td>
<td>995.7</td>
<td>1028.1</td>
<td>1017.1</td>
<td>1029.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>1004.6</td>
<td>996.1</td>
<td>1028.6</td>
<td>1022.2</td>
<td>1029.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1000</td>
<td>(1004.6)</td>
<td>(996.1)</td>
<td>1028.6</td>
<td>1022.2</td>
<td>1029.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>4000</td>
<td>(1019.3)</td>
<td>(1012.7)</td>
<td>1033.8</td>
<td>1026.0</td>
<td>1033.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>10000</td>
<td>(1055.3)</td>
<td>(1035.9)</td>
<td>1046.6</td>
<td>1038.1</td>
<td>1045.8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Pond & Pickard (1980)

Properties of Sea Water

### Table
Mechanical and thermal properties of sea water at salinity 35 g kg\(^{-1}\) and atmospheric pressure (unless otherwise stated)

<table>
<thead>
<tr>
<th>Property</th>
<th>0 °C</th>
<th>20 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic viscosity</td>
<td>(1.88 \times 10^{-3}) Pa s</td>
<td>(1.08 \times 10^{-3}) Pa s</td>
</tr>
<tr>
<td>Kinematic viscosity, (\nu)</td>
<td>(1.83 \times 10^{-6}) m(^2) s(^{-1})</td>
<td>(1.05 \times 10^{-6}) m(^2) s(^{-1})</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>(0.563) W m(^{-1}) K(^{-1})</td>
<td>(0.596) W m(^{-1}) K(^{-1})</td>
</tr>
<tr>
<td>Thermal diffusivity, (\kappa)</td>
<td>(1.37 \times 10^{-7}) m(^2) s(^{-1})</td>
<td>(1.46 \times 10^{-7}) m(^2) s(^{-1})</td>
</tr>
<tr>
<td>Prandtl number, (\nu/\kappa)</td>
<td>13.4</td>
<td>13.4</td>
</tr>
<tr>
<td>Specific heat capacity, (c_p)</td>
<td>(3985.3) J kg(^{-1}) K(^{-1})</td>
<td>(3993.3) J kg(^{-1}) K(^{-1})</td>
</tr>
<tr>
<td>Thermal expansion coefficient</td>
<td>7.2</td>
<td>7.2</td>
</tr>
<tr>
<td>Pressure = 0.1 MN m(^{-2})</td>
<td>(52 \times 10^{-8}) K(^{-1})</td>
<td>(250 \times 10^{-8}) K(^{-1})</td>
</tr>
<tr>
<td>Pressure = 100 MN m(^{-2})</td>
<td>(244 \times 10^{-8}) K(^{-1})</td>
<td>(325 \times 10^{-8}) K(^{-1})</td>
</tr>
<tr>
<td>Ratio of specific heat capacities, (c_p/\rho)</td>
<td>1.000.4</td>
<td>1.010.6</td>
</tr>
<tr>
<td>Velocity of sound, (c)</td>
<td>(1449) m s(^{-1})</td>
<td>(1522) m s(^{-1})</td>
</tr>
<tr>
<td>Compressibility</td>
<td>(4.65 \times 10^{-12}) Pa(^{-1})</td>
<td>(4.28 \times 10^{-12}) Pa(^{-1})</td>
</tr>
<tr>
<td>Freezing point</td>
<td>(-1.919) °C</td>
<td>(-1.919) °C</td>
</tr>
<tr>
<td>Boiling point</td>
<td>(370.86) °C</td>
<td>(370.86) °C</td>
</tr>
</tbody>
</table>

Kaye & Laby (2011)
Ocean dynamics

Equations of motion (I)

7 state variables:
3D velocity \((u,v,w)\), pot.temperature \((\Theta)\), salinity \((S)\), pressure \((p)\), density \((\rho)\)
require 7 equations
1.-3. Newtons 2. law -> momentum equations (3 components)

\[ a = \frac{F}{m} \iff a = \frac{\vec{F}}{\rho} = \vec{F} \]

\[ \frac{Dv}{Dt} + f(k \times v) = -\frac{1}{\rho} \nabla p - gk + \vec{F} \]

4. mass conservation -> continuity equation

\[ \frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{v} = 0 \]
Equations of motion (II)

5. equation of state
\[ \rho = R(S, T, p) \approx \rho_r [1 - \alpha(T - T_r) + \beta(S - S_r)] \]

6. conservation of energy -> equation for potential (conservative) temperature
\[ \frac{De}{Dt} = Q_e, \quad e = \rho c_p \theta, \quad \Rightarrow \frac{D\theta}{Dt} = \frac{Q_e}{\rho c_p} \]

7. conservation of salt -> equation for salinity
\[ \frac{DS}{Dt} = Q_S \propto (E - P - R) \]

Important simplifications and approximations follow:
hydrostatic balance, geostrophic balance, Boussinesq approximation

---

hydrostatic balance

3rd component of momentum equation (vertical momentum)
\[ \frac{Dw}{Dt} + [f(k \times v)]_z = \frac{-1}{\rho} \frac{\partial p}{\partial z} - g + F_z \]
\[ \Rightarrow \frac{\partial p}{\partial z} = -g \rho \]
\[ \int_{-H}^{\eta} \frac{\partial p}{\partial z} \, dz = \int_{p(-H)}^{p(\eta)} dp = p(\eta) - p_b = -g \int_{-H}^{\eta} \rho \, dz \]
\[ p_b = p(\eta) + g \int_{-H}^{\eta} \rho \, dz \]
\[ \eta = \text{dynamic topography}, \quad \text{sea level} \]
\[ p_b \approx p(\eta) + g \int_{-H}^{0} \rho \, dz + g \int_{0}^{\eta} \rho_0 \, dz = p(\eta) + g \int_{-H}^{0} \rho \, dz + g \rho_0 \eta \]
hydrostatic balance

3rd component of momentum equation (vertical momentum)

\[
\frac{Dw}{Dt} + [f(k \times v)]_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + F_z
\]
\[
\Rightarrow \frac{\partial p}{\partial z} = -g \rho
\]
\[
\int_{-H}^{\eta} \frac{\partial p}{\partial z} \, dz = \int_{p(-H)}^{p(\eta)} dp = p(\eta) - p_b = -g \int_{-H}^{\eta} \rho \, dz
\]

\(\eta\) = dynamic topography,
sea level
bottom pressure = atmospheric pressure + mass/area

\[
p_b \approx p(\eta) + g \int_{-H}^{0} \rho \, dz + g \int_{0}^{\eta} \rho_0 \, dz = p(\eta) + g \int_{-H}^{0} \rho \, dz + g \rho_0 \eta
\]

geostrophic balance

Geostrophy:
Balance between pressure gradient force and Coriolis force

1st and 2nd component of momentum equation (horizontal momentum)

\[
\frac{Du}{Dt} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} + F_x \quad \Rightarrow \quad f v = +\frac{1}{\rho} \frac{\partial p}{\partial x}
\]
\[
\frac{Dv}{Dt} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + F_y \quad \Rightarrow \quad f u = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\(f/\partial z\) and replacing the hydrostatic pressure gives the
so-called thermal or geostrophic wind equations: level-of-
no-motion problem
Geostrophy:
Balance between pressure gradient force and Coriolis force

1st and 2nd component of momentum equation (horizontal momentum)

\[
\frac{Du}{Dt} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} + F_x \quad \Rightarrow \quad fv = +\frac{1}{\rho} \frac{\partial p}{\partial x}
\]

\[
\frac{Dv}{Dt} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + F_y \quad \Rightarrow \quad fu = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\[
\frac{\partial v}{\partial z} = +\frac{1}{\rho} \frac{\partial^2 p}{\partial z \partial x} \quad \Rightarrow \quad \frac{\partial v}{\partial z} = -\frac{g}{\rho f} \frac{\partial \rho}{\partial x}
\]

\[
\frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial z \partial y} \quad \Rightarrow \quad \frac{\partial u}{\partial z} = +\frac{g}{\rho f} \frac{\partial \rho}{\partial y}
\]

\[
\Rightarrow v(z) = v_0 - \int_{z_0}^{z} \frac{g}{\rho f} \frac{\partial \rho}{\partial x} \, dz'
\]

\[
\partial / \partial z \text{ and replacing the hydrostatic pressure gives the so-called thermal or geostrophic wind equations: level-of-no-motion problem}
\]

\[
\partial / \partial z \text{ and replacing the hydrostatic pressure gives the so-called thermal or geostrophic wind equations: level-of-no-motion problem}
\]

\[
\Rightarrow v(z) = v_0 - \int_{z_0}^{z} \frac{g}{\rho f} \frac{\partial \rho}{\partial x} \, dz'
\]
alternative to thermal wind: dynamic method

classical method for inferring horizontal velocity normal to “hydrographic sections” (equivalent to thermal wind), see e.g. Gill (1980)

geopotential height: \( \Phi = g z \)

insert in hydrostatic balance:

\[-\rho g dz = -\rho d\Phi = dp \iff d\Phi = -\frac{dp}{\rho} = -v_s dp \]

\[ \Rightarrow \quad fv = \frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x} \]

\[-fu = \frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x} \]

geopotential height anomaly: \( -\Phi'(p) = \int_0^p \left( \frac{1}{\rho(S, T, p')} - \frac{1}{\rho(35, 0, p')} \right) dp' \)

\[ \Rightarrow \quad f \left\{ v(p) - v(0) \right\} = -\frac{\partial \Phi'(p)}{\partial x} \Rightarrow f \left\{ v(p_1) - v(p_2) \right\} = -\frac{\partial}{\partial x} \left\{ \Phi'(p_1) - \Phi'(p_2) \right\} \]

\[-f \left\{ u(p) - u(0) \right\} = -\frac{\partial \Phi'(p)}{\partial y} \Rightarrow f \left\{ u(p_1) - u(p_2) \right\} = -\frac{\partial}{\partial y} \left\{ \Phi'(p_1) - \Phi'(p_2) \right\} \]
1st and 2nd component of momentum equation (horizontal momentum)

\[
\begin{align*}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + F_x \quad \Rightarrow \quad f v &= +\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + F_y \quad \Rightarrow \quad f u &= -\frac{1}{\rho} \frac{\partial p}{\partial y}
\end{align*}
\]

Integrating the hydrostatic pressure again gives:

\[
\int_0^\eta \frac{\partial p}{\partial z} \, dz = \int_{p(0)}^{p(\eta)} dp = - \int_0^\eta g \rho \, dz
\]

\[
p(\eta) - p(0) = -g \rho \eta
\]

\[
\Rightarrow u = -\frac{g}{f} \frac{\partial \eta}{\partial y}
\]

\[
v = +\frac{g}{f} \frac{\partial \eta}{\partial x}
\]
geostrophic balance and surface elevation

1st and 2nd component of momentum equation (horizontal momentum)

\[
\begin{align*}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + F_x \quad \Rightarrow \quad fv = +\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + F_y \quad \Rightarrow \quad fu = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\end{align*}
\]

integrating the hydrostatic pressure again gives:

\[
\begin{align*}
\int_0^\eta \frac{\partial p}{\partial z} \, dz &= \int_{p(0)}^{p(\eta)} dp = -\int_0^\eta g \rho \, dz \\
p(\eta) - p(0) &= -g \rho \eta \\
\Rightarrow u &= -\frac{g}{f} \frac{\partial \eta}{\partial y} \\
v &= +\frac{g}{f} \frac{\partial \eta}{\partial x}
\end{align*}
\]

\[\eta = \text{dynamic topography, sea level}\]
geostrophic balance and surface elevation

1st and 2nd component of momentum equation (horizontal momentum)

\[
\frac{Du}{Dt} - f v = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \mathbf{F}_x \quad \Rightarrow \quad f v = + \frac{1}{\rho} \frac{\partial p}{\partial x}
\]

\[
\frac{Dv}{Dt} + f u = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \mathbf{F}_y \quad \Rightarrow \quad f u = - \frac{1}{\rho} \frac{\partial p}{\partial y}
\]

integrating the hydrostatic pressure again gives:

\[
\frac{\partial p}{\partial z} \, dz = \int_{p(0)}^{p(\eta)} dp = - \int_0^\eta \! \! g \rho \, dz
\]

\[
\text{Inverted Barometer} \quad p(\eta) - p(0) = - g \rho \eta
\]

\[\eta = \text{dynamic topography, sea level}\]

\[
\Rightarrow u = - \frac{g}{f} \frac{\partial \eta}{\partial y}
\]

\[v = + \frac{g}{f} \frac{\partial \eta}{\partial x}\]

Boussinesq Approximation

According to Spiegel and Veronis (1960):

1. The fluctuations in density which appear with the advent of motion result principally from thermal (as opposed to pressure) effects.

2. In the equations for the rate of change of momentum and mass, density variations may be neglected except when they are coupled to the gravitational acceleration in the buoyancy force.

\[1. \quad \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v} = 0 \Rightarrow \nabla \cdot \mathbf{v} = 0 \quad \text{mass balance becomes volume balance}\]

\[2. \quad \frac{D\mathbf{v}}{Dt} + f (k \times \mathbf{v}) = - \frac{1}{\rho_0} \nabla p - g k + \mathbf{F}\]
Consequences of the Boussinesq Approximation

Integrating the continuity equation vertically gives:

\[
\int_{-H}^{n} \left( \frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{v} \right) dz = Q_{\text{FW}}
\]

\[
\int_{-H}^{n} \nabla_h \cdot \mathbf{u} dz + \int_{-H}^{n} \frac{\partial w}{\partial z} dz = Q_{\text{FW}} - \int_{-H}^{n} \frac{1}{\rho} \frac{D\rho}{Dt} dz
\]

\[
\int_{-H}^{n} \frac{\partial w}{\partial z} dz = w(\eta) - w(-H)
\]

\[
w(\eta) = \frac{dn}{dt} = Q_{\text{FW}} - \nabla_h \cdot \int_{-H}^{n} \mathbf{u} dz + w(-H)
\]

\[
\frac{dn}{dt} = Q_{\text{FW}} - \nabla_h \cdot \int_{-H}^{n} \mathbf{u} dz + w(-H) + \int_{-H}^{n} \frac{\rho_c D(\alpha T)}{\rho} \frac{D\rho}{Dt} dz - \int_{-H}^{n} \frac{\rho_c D(\beta S)}{\rho} \frac{D\rho}{Dt} dz
\]

horizontal integral: \(\frac{\partial n}{\partial t} = Q_{\text{FW}} + \int_{-H}^{n} \frac{\rho_c D(\alpha T)}{\rho} \frac{D\rho}{Dt} dz - \int_{-H}^{n} \frac{\rho_c D(\beta S)}{\rho} \frac{D\rho}{Dt} dz\)
Consequences of the Boussinesq Approximation

Integrating the continuity equation vertically gives:

\[
\begin{align*}
\int_{-H}^{\eta} \left( \frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{v} \right) \, dz &= Q_{FW} \\
\int_{-H}^{\eta} \nabla_h \cdot \mathbf{u} \, dz + \int_{-H}^{\eta} \frac{\partial w}{\partial z} \, dz &= Q_{FW} - \int_{-H}^{\eta} \frac{1}{\rho} \frac{D\rho}{Dt} \, dz \\
\int_{-H}^{\eta} \frac{\partial w}{\partial z} \, dz &= w(\eta) - w(-H) \quad Q_{FW} - \nabla_h \cdot \int_{-H}^{\eta} \mathbf{u} \, dz - \int_{-H}^{\eta} \frac{1}{\rho} \frac{D\rho}{Dt} \, dz \\
\end{align*}
\]

horizontal integral: \( \frac{\partial \eta}{\partial t} = Q_{FW} + \int_{-H}^{\eta} \frac{\rho_e}{\rho} \frac{D(\alpha T)}{Dt} \, dz - \int_{-H}^{\eta} \frac{\rho_e}{\rho} \frac{D(\beta S)}{Dt} \, dz \)
Consequences of the Boussinesq Approximation

Integrating the continuity equation vertically gives:

\[ \int_{-H}^{H} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right) + \nabla \cdot \mathbf{v} \right) \, dz = Q_{FW} \]
\[ \int_{-H}^{H} \nabla_h \cdot \mathbf{u} \, dz + \int_{-H}^{H} \frac{\partial w}{\partial z} \, dz = Q_{FW} - \int_{-H}^{H} \frac{1}{\rho} \frac{\partial \rho}{\partial t} \, dz \]
\[ \int_{-H}^{H} \frac{\partial w}{\partial z} \, dz = w(\eta) - w(-H) \]
\[ w(\eta) = \frac{d\eta}{dt} \]
\[ \frac{d\eta}{dt} = Q_{FW} - \nabla_h \cdot \int_{-H}^{H} \mathbf{u} \, dz + w(-H) \]

horizontal integral: \[ \frac{\partial \eta}{\partial t} = Q_{FW} + \int_{-H}^{H} \frac{\rho_c}{\rho} \frac{D(\alpha T)}{Dt} \, dz - \int_{-H}^{H} \frac{\rho_c}{\rho} \frac{D(\beta S)}{Dt} \, dz \]

Consequences of the Boussinesq Approximation

Integrating the continuity equation vertically gives:

\[ \int_{-H}^{H} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right) + \nabla \cdot \mathbf{v} \right) \, dz = Q_{FW} \]
\[ \int_{-H}^{H} \nabla_h \cdot \mathbf{u} \, dz + \int_{-H}^{H} \frac{\partial w}{\partial z} \, dz = Q_{FW} - \int_{-H}^{H} \frac{1}{\rho} \frac{\partial \rho}{\partial t} \, dz \]
\[ \int_{-H}^{H} \frac{\partial w}{\partial z} \, dz = w(\eta) - w(-H) \]
\[ w(\eta) = \frac{d\eta}{dt} \]
\[ \frac{d\eta}{dt} = Q_{FW} - \nabla_h \cdot \int_{-H}^{H} \mathbf{u} \, dz + w(-H) \]

horizontal integral: \[ \frac{\partial \eta}{\partial t} = Q_{FW} + \int_{-H}^{H} \frac{\rho_c}{\rho} \frac{D(\alpha T)}{Dt} \, dz - \int_{-H}^{H} \frac{\rho_c}{\rho} \frac{D(\beta S)}{Dt} \, dz \]
Consequences of the Boussinesq Approximation

A. A Compressible Ocean

- a) Initial
- b) Adjusting
- c) Pressure

example of Boussinesq effect: geostrophic adjustment after surface heating, Huang and Jin (2002)

B. A Boussinesq Ocean

- d) Initial
- e) Adjusting
- f) Pressure

Surface wind forcing

Surface winds exert stress (τ) on ocean surface and drive a thin (order 100 m) surface layer with a lot of turbulence -> turbulent stress divergence \( \partial \sigma / \partial z \)

\[
\rho_0 f \left( \frac{\partial w G}{\partial z} \right) = \frac{\tau}{\rho_0 f}
\]

\( d \tau \)

Correlations between OBP and WSC, but note assumption of stationarity: \( d/dt = 0 \) => only long/seasonal timescales can be considered, e.g. Gill and Niiler, 1973 (DSR).
Surface wind forcing

Surface winds exert stress (τ) on ocean surface and drive a thin (order 100 m) surface layer with a lot of turbulence -> turbulent stress divergence \( \partial \sigma / \partial z \)

\[
\rho_0 f (k \times \mathbf{u}_E) + \rho_0 f (k \times \mathbf{u}_G) = -\nabla_h p + \frac{\partial \sigma}{\partial z}.
\]

with \( \nabla_h \mathbf{u}_E + \frac{\partial w_E}{\partial z} = 0 \Rightarrow w_E(0) = -\nabla_h \int_{-h_m}^{0} \mathbf{u}_E \, dz \)

\[
\nabla_h \int_{-h_m}^{0} \mathbf{u}_E \, dz = -\nabla_h \left( k \times \frac{\tau}{\rho_0 f} \right) = \text{curl} \frac{\tau}{\rho_0 f}
\]

leads to Ekman transport, Ekman pumping and mass redistribution, e.g. Song and Zlotnicki, 2008, Chambers and Willis, 2008 find correlations between OBP and WSC, but note assumption of stationarity: \( d/dt = 0 \Rightarrow \) only long/seasonal timescales can be considered, e.g. Gill and Niiler, 1973 (DSR).

---

Short-term Non-Tidal Variability: The ocean's response to atmospheric forcing
- adjustment of density field slow wrt. seasonal time-scales
- bottom pressure changes related to changes in wind stress curl

Chambers (OSD, 2011)

Wunsch and Stammer (1997)

IB assumption:

ocean is assumed to react iso-statically to changes in surface pressure

\[ \eta(t) = -\frac{p'_a(t)}{\rho_0 g} \]

\[ p'_a(t) = p_a(t) - \int \int_{\text{Ocean}} p_a(\theta, \lambda, t) \, d\theta \, d\lambda \]

IB assumption is justified away from the tropics and on time-scales longer than a few days

Wunsch and Stammer (1997)
Atmospheric Tides and the Ocean’s Response

- daily varying solar insolation is absorbed by water vapor and ozone
- causes variations in temperature, winds, and consequently surface pressure

short periods:
daily, twice-daily

ocean response deviates substantially from IB assumption

Dobslaw and Thomas (2005)

Freshwater Fluxes

Evaporation
Precipitation

River runoff

GRDC; HOAPS, MPI-M & Uni HH
variations in total ocean mass are predominantly seasonal:

Rietbroek et al. (2008)

Dobslaw et al. (2010)
Ocean Tides

Tide Generating Potential

gravitational potential \( V \) exerted by the moon:

\[
V = \frac{GM}{\rho} = \frac{GM}{R} \sqrt{1 + \left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\cos \alpha}
\]

\[
V = \frac{GM}{R} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^n P_n(\cos \alpha)
\]

tidal potential \( \Gamma \):

\[
\Gamma = V - \frac{GM}{R} - \frac{GM}{R^2} a \cos \alpha
\]

\[
\Gamma = \frac{GM}{R} \sum_{n=2}^{\infty} \left(\frac{a}{R}\right)^n P_n(\cos \alpha)
\]

Agnew (2009)
Tide Generating Potential

consider degree 2 only:
\[ \Gamma = \frac{GM}{R} \left( \frac{a}{R} \right)^2 \frac{1}{2} \left( 3 \cos^2 \alpha - 1 \right) \]

express solid angle \( \alpha \) in equatorial coordinates of \( O \) and \( M \):
\[ \Gamma = \frac{3GM}{2R} \left( \frac{a}{R} \right)^2 \left[ \cos^2 \theta \cos^2 \Theta + \frac{1}{2} \sin^2 \theta \sin^2 \Theta - \frac{1}{3} \right. \]
\[ + \frac{1}{2} \sin 2\theta \sin 2\Theta \cos(\Phi - \phi - \Omega t) \]
\[ + \frac{1}{2} \sin^2 \theta \sin^2 \Theta \cos 2(\Phi - \phi - \Omega t) \] long-period
diurnal
semi-diurnal
time dependency remains in \( R, \Theta, \Phi \)

Ephemerides of Moon and Sun

Nutation Series based on Delauney Arguments:
\[ \Delta \psi = \sum_{i=1}^{106} \Delta \psi_i = \sum_{i=1}^{106} \left( a_{0i} + a_{1i} (T - T_0) \right) \cdot \sin L ; \]
\[ \Delta \epsilon = \sum_{i=1}^{106} \Delta \epsilon_i = \sum_{i=1}^{106} \left( b_{0i} + b_{1i} (T - T_0) \right) \cdot \cos L , \]
\[ L = k_{1i} \cdot \tau + k_{2i} \cdot \tau^2 + k_{3i} \cdot F + k_{4i} \cdot D + k_{5i} \cdot \Omega \]

I  mean anomaly Moon  (anomalistic month: 27d 13h 19min)
I' mean anomaly Sun  (anomalistic year: 365d 06h 14min)
F  difference between ascending node of the mean orbit and mean longitude of the moon  ( draconitic month: 27d 05h 06min)
D  mean elongation of the moon wrt. the sun  (synodic month: 29d 12h 44min)
Omega longitude of the ascending node of the mean orbit  (18.6 years)
Harmonic Expansion of the Tidal Potential

<table>
<thead>
<tr>
<th>Tide</th>
<th>Period (hours)</th>
<th>Fundamental Argument</th>
<th>Phase (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S2</td>
<td>12.00</td>
<td>0 0 -2 2 -2 2</td>
<td>0</td>
</tr>
<tr>
<td>M2</td>
<td>12.42</td>
<td>0 0 -2 0 -2 2</td>
<td>0</td>
</tr>
<tr>
<td>M4</td>
<td>12.66</td>
<td>-1 0 0 0 -2 2</td>
<td>0</td>
</tr>
<tr>
<td>Diurnal</td>
<td>23.93</td>
<td>0 0 0 0 0 i</td>
<td>+90.0</td>
</tr>
<tr>
<td>K1</td>
<td>24.07</td>
<td>0 0 -2 2 -2 1</td>
<td>-90.0</td>
</tr>
<tr>
<td>O1</td>
<td>25.83</td>
<td>0 0 -3 0 -2 1</td>
<td>-90.0</td>
</tr>
<tr>
<td>Long Period</td>
<td>13.63</td>
<td>0 2 2 0 1 0</td>
<td>0</td>
</tr>
<tr>
<td>Mf</td>
<td>23.66</td>
<td>0 0 2 0 2 0</td>
<td>0</td>
</tr>
<tr>
<td>Mn</td>
<td>27.54</td>
<td>1 0 0 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>Ssa</td>
<td>182.62</td>
<td>0 2 -2 0 2</td>
<td>0</td>
</tr>
</tbody>
</table>


Equations of Ocean Tidal Dynamics

Laplace Tidal Equations (LTE):

$$\frac{\partial \eta}{\partial t} + \mathbf{f} \times \mathbf{v} = -g \nabla (\eta - \eta_{EQ} - \eta_{SAL}) - F$$

$$\eta_{EQ} = \frac{(1 + k_2 - h_2) \Gamma}{g}$$

$$\eta_{SAL} = \frac{3 \rho_0}{2 \rho_e} \sum_n \frac{1}{2n+1} (1 + k'_n - h'_n) \eta_n$$

more general formulations include non-linear terms and stratification (see, e.g. Zahel, 1986)

analytical solutions only available for simplified geometries: hemispheric ocean, rectangular channel, etc.

Ray (1998)
Laplace Tidal Equations (LTE):

\[ \frac{\partial \eta}{\partial t} + f \times v = -g \nabla \eta \left( \eta_{EQ} - \eta_{SAL} \right) - \mathcal{F} \]

\[ \eta_{EQ} = \frac{(1 + k_2 - h_2) \Gamma}{g} \]

\[ \eta_{SAL} = \frac{3\rho_0}{2\rho_c} \sum_n \frac{1}{2n + 1} (1 + k_n' - \frac{h_n'}{h_n}) \eta_n \]

more general formulations include non-linear terms and stratification (see, e.g. Zahel, 1986)

analytical solutions only available for simplified geometries: hemispheric ocean, rectangular channel, etc.

Ocean Normal Modes

numerical solutions of LTE depend on the resonance characteristics of the ocean basins:

(a) gravitational modes

(b1) planetary vorticity modes

(b2) topographic vorticity modes
Tide Models: (a) (semi-)empirical models: EOT08ag

- analysis of residual tidal signals in T/P and GRACE
- Bosch et al. (2009), Mayer-Gürr et al. (accepted)

Tide Models: (b) hydrodynamic w/o assimilation: TiME

- finite difference scheme
- 5° spatial resolution (2-10 km)
- time-stepping model
- forced by complete luni-solar tidal potential (Thomas, 2002)

- impact of bathymetry
- transport ellipses (5°)
- compound tides

Philipp Weis (2006), Weis et al. (2008)
finite element model
assimilates T/P altimetry by means of representer method (Bennet, 1990)

K1 (<100 cm)  
M2 (<120 cm)

Summary on Short-Term Variability
1. gravitational tides: ~ 80 hPa

2. atmospheric pressure: ~ 4 hPa
1. gravitational tides: ~ 80 hPa
2. atmospheric pressure: ~ 4 hPa
3. pressure tides: ~ 6 hPa
4. wind stress: ~ 8 hPa
1. Gravitational tides: ~ 80 hPa
2. Atmospheric pressure: ~ 4 hPa
3. Pressure tides: ~ 6 hPa
4. Wind stress: ~ 8 hPa
5. Total ocean mass: ~ 0.2 hPa

Dobslaw (2007)

1. Latent heat: ~ 1.5 cm

Dobslaw (2007)
OMCT variability: sea surface height (30 days low-pass)

1. latent heat: ~ 1.5 cm
2. net radiation: ~ 1.5 cm
3. sensible heat: ~ 1.5 cm

Dobslaw (2007)
OMCT variability: sea surface height (30 days low-pass)

1. latent heat: ~ 1.5 cm
2. net radiation: ~ 1.5 cm
3. sensible heat: ~ 1.5 cm
4. precipitation - evaporation: ~ 1.5 cm
5. continental runoff: ~ 1.5 cm

Dobslaw (2007)
De-Aliasing

...But what about variability without decent frequency characteristics?

Topex/Poseidon
\( P_{\text{orb}} = 9.9156 \text{ d} \)

\[
\Delta \phi_{\text{tide}} = \frac{2\pi P_{\text{orb}}}{T_{\text{tide}}}
\]

\[
T_{\text{alias}} = \frac{2\pi P_{\text{orb}}}{|\Delta \phi_{\text{tide}}|}
\]

<table>
<thead>
<tr>
<th>Tide</th>
<th>Period (hr)</th>
<th>( T_{\text{alias}} ) (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_2 )</td>
<td>12.420501</td>
<td>62.11</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>12.009300</td>
<td>58.74</td>
</tr>
<tr>
<td>( N_2 )</td>
<td>12.658348</td>
<td>49.53</td>
</tr>
<tr>
<td>( K_1 )</td>
<td>23.93447</td>
<td>173.10</td>
</tr>
<tr>
<td>( O_1 )</td>
<td>25.819342</td>
<td>45.71</td>
</tr>
<tr>
<td>( P_1 )</td>
<td>24.06589</td>
<td>88.89</td>
</tr>
</tbody>
</table>

Chelton et al. (2001)
Predictions of non-tidal variability at high temporal resolution (3-6 h).

- finite element barotropic ocean model forced by 6 hourly ECMWF wind and surface pressure

Dynamic Atmosphere Correction (DAC) for Satellite Altimetry (AVISO):

- Mog2D-G for high frequencies (< 20 days)
- low frequencies (>20 days): IB model, static response of the ocean to pressure, no consideration of wind effects

http://www.aviso.oceanobs.com

F. Lyard; P. Gegout et al. (2009)

De-aliasing for Satellite Altimetry: Mog2D/T-UGO

De-aliasing for Satellite Gravimetry: OMCT

- 1.875° regular grid, 13 z-layers, 30 min
- forced by wind, pressure, freshwater, heat
- optional: tides, river runoff from LSDM, SAL

AOD1B RL04:

- no tides, no river runoff
- total ocean mass constant
- forcing from ECMWF 6h analyses

AOD1B RL05alpha:

- 1.0° regular grid, 20 z-layers, 20 min
- forcing as for RL04

Dobslaw and Thomas (2007)
GRACE product philosophy: GSM and GAx

GAA: Atmosphere (3D)  
GAB: Ocean

Atmosphere (3D) + Ocean
GAC = GAA + GAB

Ocean Bottom Pressure
GAD = GAB + atm. surface pressure @ wet points

GSM: variability without Atmosphere and Ocean

GSM+GAC: variability including Atm. and Ocean

Flechtner (2007)
Beyond De-aliasing: Kalman Smoother Approach

- Kurtenbach et al. (2009): consideration of correlations between daily snap-shots of the gravity field
- auto-covariances have been derived empirically from numerical models of water storage variations
- extension of the approach to include atmosphere and ocean as well (AOD products applied only for sub-daily variability)

OBP variability ACC area: 10-30 days band pass

ITG-Kalman (daily)
CNES/GRGS (10 days)
• ocean dynamics are generally described by seven equations: (momentum, continuity, conservation equations for heat and salt)

• approximations are frequently applied in OGCMs: hydrostatic, Boussinesq -> be aware of their implications!

• sea-level (and bottom pressure) variability are dominated by ocean tides and non-tidal variability (red spectrum)

• selected geophysical signals are removed from observations before averaging: tides, atmospheric pressure effects, response to winds, etc. -> be aware of the way these signals have been removed!

Textbooks


Requirements for future satellite missions

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SPP1257 Sommerschule – Globaler Wasserkreislauf
Mayschoß/Ahrtal
12.-16. September 2011

• future satellite gravimetry missions
• “connector“: spectral connection of earth interior to its exterior
• its derivation
• its interpretation
• its relationship to geophysical mass processes
• its use for simple estimates
The generally agreed wish list:
- continuation of GRACE time series: discover changes
- higher spatial resolution (from 400km to 200km or even 100km)
- no striation → less filtering → better mass estimates
- no aliasing: from ocean tides, atmosphere and ocean
- higher accuracy: better mass estimates
- series of missions

Point of departure:
- a successful CHAMP, GRACE and GOCE
### Summary

<table>
<thead>
<tr>
<th>Science Requirements (monthly time resolution)</th>
<th>Technical Requirements needed to reach science requirements (for satellite distance 200 km, height 373 km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spatial Resolution</td>
<td>Laser Interferometer coloured noise with white noise level of: Accelerometer coloured noise with white noise level of:</td>
</tr>
<tr>
<td>400 km</td>
<td>Laser Interferometer coloured noise with white noise level of: Accelerometer coloured noise with white noise level of:</td>
</tr>
<tr>
<td>1 cm</td>
<td>50 nm/√Hz</td>
</tr>
<tr>
<td>1 mm/y</td>
<td>1 mm/y</td>
</tr>
<tr>
<td>1 mm</td>
<td>0.1 mm</td>
</tr>
<tr>
<td>200 km</td>
<td>10 cm</td>
</tr>
<tr>
<td>1 cm/y</td>
<td>0.1 mm/y</td>
</tr>
<tr>
<td>1 cm</td>
<td>1 mm/y</td>
</tr>
<tr>
<td>1 mm/y</td>
<td>0.1 mm/y</td>
</tr>
</tbody>
</table>

from: proposal e-motion to ESA, 2010

### Summary

<table>
<thead>
<tr>
<th>Mission Parameter</th>
<th>e.motion Proposal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observation Concept</td>
<td>Satellite-to-Satellite Tracking in low-low mode: Observation is range (range-rate) between two low flying satellites.</td>
</tr>
<tr>
<td>Mission Duration</td>
<td>Nominal 7 years (plus possible extension).</td>
</tr>
<tr>
<td>Inclination</td>
<td>Polar or near-polar.</td>
</tr>
<tr>
<td>Repeat Cycle</td>
<td>Near monthly repeat cycle (with sub-cycle of about 10 days)</td>
</tr>
<tr>
<td>Orbit Height</td>
<td>373 km</td>
</tr>
<tr>
<td>Mission Configuration</td>
<td>Pendulum orbit with slightly rotated orbit planes (relative between the 2 satellites)</td>
</tr>
</tbody>
</table>

from: proposal e-motion to ESA, 2010
Signal amplitudes of mass variations in EWH as a function of spatial resolution, together with present-day and e-motion performance and resolution. Solid Earth mass variations are converted to EWH. Contributions from seasonal to inter-annual variations (left panel), and contributions from long-term trends (right panel).

**Summary**

<table>
<thead>
<tr>
<th>Research objectives</th>
<th>Time scales</th>
<th>Expected signals (EWH, Geoid, g)</th>
<th>Precision, resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continental water storage variations</td>
<td>weeks to decades</td>
<td>several dm EWH</td>
<td>1 cm EWH @ 400 km, 10 cm EWH @ 200 km, 1 mm EWH/y @ 400 km</td>
</tr>
<tr>
<td>Ice sheets mass balance</td>
<td>months to decades</td>
<td>dm to m EWH</td>
<td>1 cm EWH @ 400 km, 10 cm EWH @ 200 km, 1 mm EWH/y @ 400 km</td>
</tr>
<tr>
<td>Oceanic mass variations</td>
<td>hours to decades</td>
<td>cm to dm EWH</td>
<td>5 mm EWH @ 500 km, 1 mm EWH / y</td>
</tr>
<tr>
<td>GIA</td>
<td>secular to decades</td>
<td>2 mm geoid/y</td>
<td>0.01 mm geoid/y @ 400 km</td>
</tr>
<tr>
<td>Earthquakes (Mw 7-8)</td>
<td>instantaneous to centuries</td>
<td>0.1 to 1 mm geoid or 5 μGal</td>
<td>0.1 μGal @ 200 km or 0.1 mm geoid @ 400 km</td>
</tr>
<tr>
<td>Earthquakes (Mw 7-8)</td>
<td>decades to centuries</td>
<td>0.01 to 0.1 mm geoid / y or 0.5 μGal/yr</td>
<td>0.01 μGal /y @ 200 km or 0.01 mm /y geoid @ 400 km</td>
</tr>
<tr>
<td>Mantle convection &amp; plate tectonics</td>
<td>decades to centuries</td>
<td>0.05 mm geoid / yr</td>
<td>0.01 mm geoid /yr @ 400 km</td>
</tr>
<tr>
<td>Height reference, orbits, etc.</td>
<td>hours to decades</td>
<td>few cm geoid</td>
<td>1 mm geoid @ 200 km, 1 μGal @ 200 km</td>
</tr>
</tbody>
</table>

1 cm EWH in a spherical cap of radius 2000 km (800 km, 400 km, 200 km, 100 km respectively) maps to a 0.5 mm amplitude geoid variation (0.3 mm, 0.15 mm, 0.08 mm, 0.04 mm resp.).

from: proposal e-motion to ESA, 2010
next generation satellite gravimetry
any proposal for a
next generation satellite gravimetry mission
will be based on a comprehensive end-to-end simulation,
using the actual requirements in earth sciences
(e.g. essential climate variables, ECVs)
and the heritage of CHAMP, GRACE and GOCE

thereby a key formula is

$$\begin{align*}
\{\Delta C_{nm}\} &= \frac{3}{4\pi a^2} \frac{1+k_n}{2n+1} \int \Delta p(\theta, \lambda, r) \left( \frac{r}{a} \right)^{n+2} \ sin m \lambda \ \cos \theta d\theta \\
\{\Delta \bar{S}_{nm}\} &= \int \bar{P}_{nm} \cos \theta d\theta
\end{align*}$$

my lecture tries to be a summary of

Wahr, Molenaar & Bryan: Time variability of the Earth’s gravity field: Hydrological and oceanic effects and their possible detection using GRACE, JGR, 1998

earth exterior

(residual) gravitational potential outside and on the earth surface:

$$\delta V_p = \frac{GM}{a} \sum_{n=0}^{n_{\text{max}}} \left( \frac{a}{r_p} \right)^{n+1} \sum_{m=0}^{n} (\Delta C_{nm} \ cos m \lambda_p + \Delta \bar{S}_{nm} \ sin m \lambda) \bar{P}_{nm} (\cos \theta)$$

follows from

$Lap \delta V = 0$ for $r_p > a$

and boundary values

example: (residual) geoid heights:

$$\delta N_p = a \sum_{n=0}^{n_{\text{max}}} \sum_{m=0}^{n} (\Delta C_{nm} \ cos m \lambda_p + \Delta \bar{S}_{nm} \ sin m \lambda) \bar{P}_{nm} (\cos \theta)$$

Conclusion: no need to think about earth interior, densities, etc.
connection of earth exterior and interior

It is based on Newton’s law of gravitation:

\[ \delta V_p = G \iiint_{\Sigma} \frac{\Delta \rho_Q}{\ell_{PQ}} d\Sigma_Q \]

It connects the exterior („P“) with the interior („Q“).

The connector is:

\[ \ell_{PQ} \]

For integration P and Q have to be separated.

The most important formula here:

\[
\frac{\Delta C_{nm}}{\Delta S_{nm}} = \frac{3}{4\pi a^2} \frac{1}{2n+1} \int_0^1 \left( \int_0^{\pi/2} \Delta \rho(\theta, \lambda, r_0) \left( \frac{r_0}{a} \right)^{n+2} dr_0 \right) P_{nm}(\cos \theta_0) \left\{ \begin{array}{c} \cos m\lambda_0 \\ \sin m\lambda_0 \end{array} \right\} ds_{\theta_0}
\]

Why?

• It delivers the \( \Delta C_{nm} \) and \( \Delta S_{nm} \)
• It connects to densities
• It is the starting point of end-to-end simulations

But

• where does it come from,
• what do all terms mean and
• how can it be used for simple examples?
connection of earth exterior and interior

\[ \delta V_p = G \iiint_{\ell P_0} \Delta \rho \, d\Sigma_0 \]

(1) expansion of the reciprocal distance into Legendre polynomials

\[
\frac{1}{\ell P_0} = \frac{1}{r_P} \sum_{n=0}^{\infty} \left( \frac{r_Q}{r_P} \right)^n \ell P_0 \cos \psi_{\ell P_0} \quad \text{and} \quad r_Q < r_P
\]

(2) addition theorem of Legendre functions:

\[
P_n(\cos \psi_{\ell P_0}) = \frac{1}{2n+1} \sum_{m=0}^{n} \left[ \cos m \lambda_p \cdot \cos m \lambda_Q + \sin m \lambda_p \cdot \sin m \lambda_Q \right] \cdot \\
\ell P_{nm}(\cos \theta_p) \cdot \ell P_{nm}(\cos \theta_Q)
\]

connection of earth exterior and interior

\[ \delta V_p = \frac{GM}{a} \sum_{n=0}^{\infty} \left( \frac{a}{r_P} \right)^{n+1} \sum_{m=0}^{n} \left\{ \frac{1}{M(2n+1)} \int_{r_Q}^{r_P} \Delta \rho \ell P_{nm}(\cos \theta_Q) \cos m \lambda_Q d\Sigma_0 \right\} \cos m \lambda_p \ell P_{nm}(\cos \theta_p) + \\
\frac{1}{M(2n+1)} \int_{r_Q}^{r_P} \Delta \rho \ell P_{nm}(\cos \theta_Q) \cos m \lambda_Q d\Sigma_0 \right\} \sin m \lambda_p \ell P_{nm}(\cos \theta_p)
\]

Or divided into two parts:

\[ \delta V_p = \frac{GM}{a} \sum_{n=0}^{\infty} \left( \frac{a}{r_P} \right)^{n+1} \sum_{m=0}^{n} \left( \Delta \ell C_{nm} \cos m \lambda_p + \Delta \ell S_{nm} \sin m \lambda \right) \ell P_{nm}(\cos \theta_p) \]

and with

\[ d\Sigma = r^2 dr ds = r^2 dr \sin \theta d\theta d\lambda \]

and

\[ M = \bar{\rho} \cdot \text{Volume} = \frac{4}{3} \pi a^3 \]

\[
\left\{ \Delta \ell C_{nm}, \Delta \ell S_{nm} \right\} = \frac{3}{4 \pi \bar{\rho} a} \frac{1}{2n+1} \iiint r^2 \Delta \rho(\theta_Q, \lambda_Q, r_Q) \left( \frac{r_Q}{a} \right)^{n+2} dr_Q d\theta_Q d\lambda_Q \ell P_{nm}(\cos \theta_Q) \left\{ \cos m \lambda_Q, \sin m \lambda_Q \right\} ds_Q
\]
connection of earth exterior and interior

\[
\delta V_p = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{1}{M(2n+1)} \left[ \int_0^{\infty} \frac{r_0}{a} \Delta \rho_m \bar{P}_{nm}(\cos \theta_0) \cos m\lambda_0 d\Sigma_0 \right] \cos m\lambda_p \bar{P}_{nm}(\cos \theta_p) + \right. \\
\left. \frac{1}{M(2n+1)} \left[ \int_0^{\infty} \frac{r_0}{a} \Delta \rho_m \bar{P}_{nm}(\cos \theta_0) \cos m\lambda_0 d\Sigma_0 \right] \sin m\lambda_p \bar{P}_{nm}(\cos \theta_p) \right\}
\]

Or divided into two parts:

\[
\delta V_p = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \Delta \bar{C}_{nm} \cos m\lambda_p + \Delta \bar{S}_{nm} \sin m\lambda_p \right) \bar{P}_{nm}(\cos \theta_p)
\]

and with

\[d\Sigma = r^2 dr ds = r^2 dr \sin \theta d\theta d\lambda\]

and

\[M = \bar{\rho} \cdot \text{Volume} = \bar{\rho} \cdot \frac{4}{3} \pi a^3\]

\[
\left\{ \Delta \bar{C}_{nm} \right\} = \frac{3}{4\pi \bar{\rho}} \frac{1}{2n+1} \int_0^{\pi} \int_0^{\infty} \Delta \rho(\theta, \lambda, r_0) \left( \frac{r_0}{a} \right)^{n+2} d\theta_0 d\lambda_0 \bar{P}_{nm}(\cos \theta_0) \left\{ \cos m\lambda_0 \right\} \sin m\lambda_0 ds_0
\]

Can be written as:

\[
\left\{ \Delta \bar{C}_{nm} \right\} = \frac{3}{2n+1} \left\{ \Delta \bar{C}_{nm} \Delta \rho \right\}
\]

Can be written as:

\[
\left\{ \Delta \bar{S}_{nm} \right\} = \frac{3}{2n+1} \left\{ \Delta \bar{S}_{nm} \Delta \rho \right\}
\]

SH-potential coefficients versus SH-density coefficients

! on both sides dimensionless !
connection of earth exterior and interior

\[
\begin{align*}
& \left\{ \frac{\Delta C_{nm}}{\Delta S_{nm}} \right\} = \frac{3}{4\pi a} \frac{1}{2n+1} \sum_{s} \left[ \int_{r} \Delta \bar{p} \left( \theta_q, \lambda_q, r_q \right) \left( \frac{r_q}{a} \right)^{n+2} \right] \widehat{P}_{nm} \left( \cos \theta_q \right) \left\{ \cos m \lambda_q \left( \sin \theta_q \right) \right\} ds_q \\
& \text{Can be written as:} \quad \left\{ \frac{\Delta C_{nm}}{\Delta S_{nm}} \right\} = \frac{3}{2n+1} \left\{ \frac{\Delta C_{\Delta \rho}}{\Delta S_{\Delta \rho}} \right\}
\end{align*}
\]

SH-potential coefficients versus SH-density coefficients
! both dimensionless !

**SH-pocket guide**

of geodetic gravity functionals such as gravity anomalies, gravity gradients…

**density distribution:**
isostasy, GIA, dynamic topography, geophysical fluids (ocean, hydrology, …)

---

**interpretation**

from a radial density distribution to a surface layer:

\[
\begin{align*}
& \left\{ \frac{\Delta C_{nm}}{\Delta S_{nm}} \right\} = \frac{3}{4\pi a} \frac{1}{2n+1} \sum_{s} \left[ \int_{r} \Delta \bar{p} \left( \theta_q, \lambda_q, r_q \right) \left( \frac{r_q}{a} \right)^{n+2} \right] \widehat{P}_{nm} \left( \cos \theta_q \right) \left\{ \cos m \lambda_q \left( \sin \theta_q \right) \right\} ds_q \\
& \text{Replace integral by mass layer of constant (!) thickness } \Delta r=h: \\
& \int_{r} \Delta \bar{p} \left( \theta_q, \lambda_q, r_q \right) \left( \frac{r_q}{a} \right)^{n+2} dr_q \approx \Delta \bar{p}(\theta, \lambda) \cdot h = \Delta \sigma(\theta, \lambda) = \rho_w \cdot H^{EWH}(\theta, \lambda)
\end{align*}
\]

It holds (for \( h < a \)):

\[
\frac{1}{a^{n+2}} \int_{r=a}^{a+h} r^{n+2} dr = \frac{1}{a^{n+2}} \frac{1}{n+3} \left[ (a + h)^{n+3} - a^{n+3} \right] = \frac{a}{n+3} \left[ (1 + \frac{h}{a})^{n+3} - 1 \right] \\
= \frac{a}{n+3} \left[ (n+3) \frac{h}{a} + \frac{(n+3)(n+4)}{2} \left( \frac{h}{a} \right)^2 + \ldots \right] \approx h
\]
from pressure to surface layer to equivalent water height

\[
\begin{align*}
\Delta \rho_{\text{atm}}(\theta, \lambda) \cdot h_{\text{atm}} & \quad 
\Delta \rho_w(\theta, \lambda) \cdot h_w \\
\Delta \rho_{\text{ice}}(\theta, \lambda) \cdot h_{\text{ice}} & \quad 
\Delta \rho_{\text{rock}}(\theta, \lambda) \cdot h_{\text{rock}}
\end{align*}
\]

\[= \Delta \sigma(\theta, \lambda) = \rho_w \cdot h^{\text{EWH}}(\theta, \lambda)\]

\[
\begin{array}{c}
\Delta \rho \\
\hline
\Delta \sigma \\
\rho_w \{ h^{\text{EWH}} \}
\end{array}
\]

\[
\begin{align*}
\bar{\rho} &= 5517 \text{ kg/m}^3 \\
\rho_w &= 1000 \text{ kg/m}^3 \\
\rho_{\text{ocean}} &= 1027 \text{ kg/m}^3 \\
\rho_{\text{rock}} &= 2670 \text{ kg/m}^3
\end{align*}
\]

We can continue now to work with surface layers

\[
\begin{align*}
\Delta C_{nm} & \quad \Delta S_{nm} \\
\end{align*}
\]

\[
\int_r \left \{ \Delta \rho(\theta, \lambda, r) \left ( \frac{r}{\bar{\rho}} \right )^{n+2} \right \} \bar{p}_{\text{nn}}(\cos \theta) \left \{ \cos m \lambda r \right \} \left \{ \sin m \lambda r \right \} ds_r
\]

\[
\begin{align*}
\Delta C_{nm} & \quad \Delta S_{nm} \\
\end{align*}
\]

\[
\int_r \left \{ \Delta \sigma(\theta, \lambda) \right \} \bar{p}_{\text{nn}}(\cos \theta) \left \{ \cos m \lambda r \right \} \left \{ \sin m \lambda r \right \} ds_r
\]

or with equivalent water heights

\[
\begin{align*}
\Delta C_{nm} & \quad \Delta S_{nm} \\
\end{align*}
\]

\[
\int_r \left \{ \rho_w \right \} \left \{ h^{\text{EWH}}(\theta, \lambda) \right \} \bar{p}_{\text{nn}}(\cos \theta) \left \{ \cos m \lambda r \right \} \left \{ \sin m \lambda r \right \} ds_r
\]

- dimensionless -
interpretation

load Love number $k_n$

The surface layer represents a load on the elastic earth body

$$\begin{align*}
\begin{bmatrix}
\Delta C_{nm} \\
\Delta S_{nm}
\end{bmatrix} &= \frac{\rho_w}{\rho} \frac{3 \cdot (1 + k_n)}{2n + 1} \frac{1}{4\pi} \int_{s} h^{EWH}(\theta, \lambda) \tilde{P}_{nm}(\cos \theta_Q) \left\{ \begin{array}{c}
\cos m\lambda_Q \\
\sin m\lambda_Q
\end{array} \right\} ds_Q
\end{align*}$$

We get the combined effect:

1. the direct attraction of the surface layer and
2. the secondary (negative) effect of deformation

Assumption: perfectly elastic earth i.e. Hooke’s law

Literature:
Farrell WE: Deformation of the earth by surface loads, RevGeoph., 1972
Munk WH & GJF MacDonald: The rotation of the earth, ch.5.8, 1975
Chao BF: The geoid and earth rotation, in: Vanicek P & N Christou, 1994
Han D & J Wahr: the viscoelastic relaxation…GJI, 1995

interpretation

load Love number $k_n$

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<tr>
<th>$l$</th>
<th>$k_n$</th>
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<tbody>
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<tr>
<td>200</td>
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</table>

for $n>1$:
all Love numbers negative

Wahr, Molenaar & Bryan, JGR, 1998
SH-degree n=0

- total mass of solid earth and its fluids remains constant (mass conservation)
- M in geodesy includes: solid earth, ice, oceans and atmosphere
- mass of single fluid components (atmosphere, ocean, ..) is not constant
- variable atmospheric and oceanic mass does not cause deformation of solid earth mass \( \rightarrow k_0 = 0 \)
- from Boussinesq approximation to mass conservation

SH-degree n=1

\( \Delta \tilde{C}_{10}, \Delta \tilde{C}_{11} \text{ and } \Delta \tilde{S}_{11} \) are proportional to the position of earth centre of mass (CoM) relative to centre of coordinate system

Case 1: origin of coordinate system: centre of mass

\[ \Delta \tilde{C}_{10} = \Delta \tilde{C}_{11} = \Delta \tilde{S}_{11} = 0 \]

degree n=1 components need not to vanish, but their sum

The Love number is \( k_{n=1} = -1 \) in this case

Case 2: origin of c.s.: centre of figure of solid earth surface

degree n=1-coefficients define now the off-set between the CoM of solid earth plus deformation and the centre of the deformed figure

It holds now: \( k_{n=1} = -(h_{n=1} + 2\ell_{n=1})/3 = +0.027 \)
just for completeness

the formula and its inverse

\[
\frac{\Delta C_{nm}}{\Delta S_{nm}} = \frac{3 \cdot (1 + k_n)}{2n + 1} \frac{1}{4\pi} \int_{s} \Delta \sigma(\theta, \lambda) \bar{P}_{nm}(\cos \theta) \left\{ \begin{array}{c} \cos m\lambda_0 \\ \sin m\lambda_0 \end{array} \right\} ds_0
\]

\[
\Delta \sigma(\theta, \lambda) = a \cdot \bar{\rho} \sum_{n=0}^{n_{\text{max}}} \frac{2n + 1}{3(1 + k_n)} \sum_{m=0}^{m_{\text{max}}} \bar{P}_{nm}(\cos \theta) \left\{ \Delta \tilde{C}_{nm} \cos m\lambda + \Delta \tilde{S}_{nm} \sin m\lambda \right\}
\]

Sh-synthesis

\[
\frac{\Delta \tilde{C}_{nm}}{\Delta \tilde{S}_{nm}} = \frac{\rho_w}{\bar{\rho}} \frac{3 \cdot (1 + k_n)}{2n + 1} \frac{1}{4\pi} \int_{s} h^{EWH}(\theta, \lambda) \bar{P}_{nm}(\cos \theta) \left\{ \begin{array}{c} \cos m\lambda_0 \\ \sin m\lambda_0 \end{array} \right\} ds_0
\]

\[
h^{EWH}(\theta, \lambda) = a \cdot \bar{\rho} \sum_{n=0}^{n_{\text{max}}} \frac{2n + 1}{3(1 + k_n)} \sum_{m=0}^{m_{\text{max}}} \bar{P}_{nm}(\cos \theta) \left\{ \Delta \tilde{C}_{nm} \cos m\lambda + \Delta \tilde{S}_{nm} \sin m\lambda \right\}
\]

Chambers & Schrötter, JoGeodynamics, 2011, accepted

relationship to geophysical mass processes

<table>
<thead>
<tr>
<th>TABLE 6.1 EARTH’S WATER</th>
</tr>
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<td>Estimates given in column 2 are in millions of cubic kilometers (MClam), where a cubic kilometer is a billion cubic meters. LGM stands for the Last Glacial Maximum, 18,000 years ago. Equivalent thickness (column 4) are obtained by dividing the volumes of water by the areas concerned. Rivers, streams, and the biosphere contain minute percentages of the total water, given in column 3 in parts per million (1 ppm = 0.0001%). Estimates of the amount of water underground are very uncertain. The entries correspond to layers from the surface down to 4,000 meters depth. Shallow fresh groundwater, from the surface down to 750 meters depth, amounts to about 4 MClam, and has shorter residence time. Estimates for fresh and saline water in layers more than 4,000 meters deep, not included in the table, range from 50 to 250 MClam.</td>
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<tr>
<td>Type of Water</td>
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<tr>
<td>--------------------------</td>
</tr>
<tr>
<td>Total</td>
</tr>
<tr>
<td>Salt water (oceans, inland seas, salt lakes)</td>
</tr>
<tr>
<td>Of which, water from ice melt since LGM</td>
</tr>
<tr>
<td>Present-day “permanent” snow/glaciers</td>
</tr>
<tr>
<td>If all melts</td>
</tr>
<tr>
<td>Present-day permafrost</td>
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<tr>
<td>Soil moisture</td>
</tr>
<tr>
<td>Underground freshwater</td>
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<td>Underground saltwater</td>
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<tr>
<td>Freshwater lakes</td>
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<tr>
<td>Rivers and streams</td>
</tr>
<tr>
<td>Biosphere</td>
</tr>
<tr>
<td>Atmospheric water vapor and cloud water/ice</td>
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</tbody>
</table>

Kandel R: water from heaven, 2003
relationship to geophysical mass processes

from pressure to surface layer to equivalent water height

\[
\begin{align*}
\Delta \rho_{\text{atm}} \cdot h_{\text{atm}} \\
\Delta \rho_{\text{w}} \cdot h_{\text{w}} \\
\Delta \rho_{\text{ice}} \cdot h_{\text{ice}} \\
\Delta \rho_{\text{rock}} \cdot h_{\text{rock}}
\end{align*}
\]

\[
= \Delta \sigma(\theta, \lambda) = \rho_{\text{w}} \cdot h^{\text{EWH}}(\theta, \lambda)
\]

\[
\Delta \sigma_{\text{atm}} = \frac{\Delta \rho_{\text{atm}}}{g} = \int \Delta \rho_{\text{atm}} \, \text{d}r
\]

\[
\Delta \sigma_{\text{oce}} = \rho_{\text{oce}} \cdot \Delta \zeta_{\text{forcing}}
\]

\[
\Delta \rho_{\text{OBP}} = (\Delta \rho_{\text{atm}} + g \cdot \rho_{\text{oce}} \cdot \Delta \zeta_{\text{barometer}}) + g \cdot \rho_{\text{oce}} \cdot \Delta \zeta_{\text{forcing}} + g \cdot \int \Delta \rho_{\text{oce}} \, \text{d}r
\]

\[
\rho = 5517 \, \text{kg/m}^3 \\
\rho_{\text{w}} = 1000 \, \text{kg/m}^3 \\
\rho_{\text{ocean}} = 1027 \, \text{kg/m}^3 \\
\rho_{\text{rock}} = 2670 \, \text{kg/m}^3
\]

relationship to geophysical mass processes

atmospheric pressure profiles provided by weather services 6-hourly on a global spherical 3D grid

- atmosphere

- continental processes:
  - continental water storage
  - ice sheet mass balance
  - earthquakes (coseismic)
  - earthquakes (post-seismic)
  - volcano eruption
  - mantle convection & plates

- oceanic processes:
  - assimilation into OCMs
  - deep ocean currents
  - separation of steric and non-steric part of sea level rise

- glacial isostatic adjustment (GIA)

GIA-models depend on viscosity profile, ice loading history and polar wander
### Relationship to Geophysical Mass Processes

**Continental processes:**
Correct best possible for contributions from:

- **A)** atmospheric mass changes
  - De-aliasing using data from ECMWF

- **continental processes:**
  - continental water storage
  - ice sheet mass balance
  - earthquakes (coseismic)
  - earthquakes (post-seismic)
  - volcano eruption
  - mantle convection & plates

- **B)** ocean contribution in coastal zones:
  - tidal forcing,
  - thermal and wind forcing,
  - ocean part of global water cycle

- **C)** glacial isostatic adjustment modelling and where possible direct measurement

**Pressure of total column: ocean plus atmosphere:**

\[
\Delta P_{\text{OBP}} = (\Delta P_{\text{atm}} + g \cdot \rho_{\text{oce}} \cdot \Delta \zeta_{\text{barometer}}) + g \cdot \rho_{\text{oce}} \cdot \Delta \zeta_{\text{forcing}} + g \cdot \int \Delta P_{\text{oce}} \, dr
\]

**Oceanic processes:**

- assimilation into OCMs
- deep ocean currents
- separation of steric and non-steric part of sea level rise

**Glacial isostatic adjustment (GIA)**
Convenient test functions for load estimates

A) Circular Cylinder [H-M 1967, ch.3]: P located on symmetry axis on cylinder constant density \( \rho \), radius \( s \), height \( \Delta h \) (with \( \Delta h < s \))

Effect on gravity potential in P:
\[
\delta V = \pi G \rho \left[ -\Delta h^2 + \Delta h \sqrt{s^2 + \Delta h^2} + \ln \frac{\Delta h + \sqrt{s^2 + \Delta h^2}}{s} \right] \\
= \pi G \rho \left[ -\Delta h^2 + \Delta h \cdot s + \Delta h \cdot s \right] \\
= 2\pi G \rho \Delta h \cdot s
\]

Effect on geoid in P:
\[
\delta N = \frac{\delta V}{g} = \frac{2\pi G \rho \Delta h \cdot s}{\left( \frac{4}{3} \pi G \bar{\rho} R \right)} = \frac{3 \rho \Delta h \cdot s}{2 \bar{\rho} R}
\]

Example with \( s=1000\text{km} \):
\[
\delta N = 1.5 \cdot 0.18 \cdot 0.16 \cdot \Delta h = 0.042 \cdot \Delta h \\
\text{where } \rho \bar{\rho} / 1000 / 5517 = 0.18
\]

B) Spherical Density Layer [H-M 1967, ch.3-8]: P located on symmetry axis of spherical layer with constant surface density \( \sigma \) with radius opening angle \( \Psi \)

Effect on gravity potential:
\[
\delta V = G R^2 \sigma \int_0^{2\pi} \int_0^{\Psi} \frac{1}{\ell} \sin \psi d\psi d\alpha \\
= 2\pi G R^2 \sigma \int_0^{\Psi} \frac{\sin \psi}{2R \sin \frac{\Psi}{2}} d\psi \\
= 4\pi G \sigma R \sin \frac{\Psi}{2}
\]

Effect on geoid:
\[
\delta N = \frac{\delta V}{g} = \frac{4\pi G \sigma R \sin \frac{\Psi}{2}}{\left( \frac{4}{3} \pi G \bar{\rho} R \right)} = \frac{3 \sigma}{\bar{\rho}} \sin \frac{\Psi}{2} \\
\approx \frac{3}{2} \frac{\rho \Delta h}{\bar{\rho}} \frac{s}{R}
\]
Example: mass change with the following properties
origin located at north pole
isotropic (not dependent on longitude)
thin layer of thickness $h^{EWH}$

This results in:

$$\Delta \tilde{C} = \frac{\rho_w}{\tilde{\rho}} \frac{3 \cdot (1 + k_n)}{2n+1} \int_0^\pi \frac{h^{EWH}(\theta, \lambda)}{a} P_{nm}(\cos \theta) \left\{ \cos m \lambda \right\} \sin m \lambda \, ds_\lambda$$

Expansion of $h^{EWH}$ into a Legendre series gives:

$$\frac{1}{2} \int_0^\pi \frac{h^{EWH}(\theta)}{a} P_n(\cos \theta) d\theta = \frac{\sqrt{2n+1}}{2} \int_0^\pi \frac{h^{EWH}(\theta)}{a} P_n(\cos \theta) d\theta$$

$$= \sqrt{2n+1} \frac{h_n^{EWH}}{a} = \sqrt{2n+1} \Delta C_n^{EWH}$$

and therefore:

$$\Delta \tilde{C}_{nm} = 0 \quad \text{for} \quad m \neq 0 \quad \text{and} \quad \Delta \tilde{S}_{nm} = 0 \quad \text{and}$$

$$\Delta \tilde{C}_n = \frac{\rho_w}{\tilde{\rho}} \frac{3 \cdot (1 + k_n)}{\sqrt{2n+1} a} h^{EWH}_n = \frac{\rho_w}{\tilde{\rho}} \frac{3 \cdot (1 + k_n)}{\sqrt{2n+1} a} \Delta C_n^{EWH}$$
application of basic formula to a mass layer

Expansion into Legendre polynomials:

$$\int_{-1}^{1} P_n(t) P_k(t) dt = \frac{2}{2n + 1} \delta_{nk}$$

$$\overline{P}_n(t) = \sqrt{2n + 1} P_n(t)$$ and $$P_n(t) = \overline{P}_n(t) / \sqrt{2n + 1}$$

$$\bar{f}_n = \frac{1}{2} \int f(t) \overline{P}_n(t) dt$$ and $$f(t) = \sum \bar{f}_n \overline{P}_n(t)$$

$$f_n = \frac{2n + 1}{2} \int f(t) P_n(t) dt$$ and $$f(t) = \sum f_n P_n(t)$$
application of basic formula to a mass layer

\[ \delta V_n = \frac{GM}{a} \left( \frac{a}{r} \right)^{n+1} \Delta \bar{C}_n \quad \text{and} \quad \delta N_n = a \Delta \bar{C}_n \]

\[ \delta N_n = a \cdot \Delta \bar{C}_n = \frac{\rho_w}{\bar{\rho}} \frac{3 \cdot (1 + k_n)}{\sqrt{2n+1}} h_{n}^{EWH} = a \cdot \frac{\rho_w}{\bar{\rho}} \frac{3 \cdot (1 + k_n)}{\sqrt{2n+1}} \Delta C_{n}^{EWH} \]

convenient test functions for load estimates

C) Pellinen Function: spherical equivalent of a box function

\[ B(\psi) = \begin{cases} 1 & \text{and } \psi \leq \Psi \\ \frac{1}{2\pi(1-\cos \Psi)} & \text{and } \psi > \Psi \end{cases} \]

or as Legendre series

\[ B(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \beta_n P_n(\cos \psi) \]

Expansion into Legendre polynomials:

\[ \beta_n = 2\pi \int_{\psi=0}^{\pi} B(\psi) P_n(\cos \Psi) \sin \psi d\psi = \frac{1}{1-\cos \Psi} \int_{\psi=0}^{\pi} P_n(\cos \Psi) \sin \psi d\psi \]

\[ = \frac{1}{1-\cos \Psi} \frac{1}{2n+1} \left[ P_{n-1}(\cos \Psi) - P_{n+1}(\cos \Psi) \right] \]

Or approximately (Sjöberg L.B.G., 1980):

\[ \beta_n = \frac{2n-1}{n+1} \cos \Psi \cdot \beta_{n-1} - \frac{n-2}{n+1} \beta_{n-2} \quad \text{with } \beta_0 = 1 \text{ and } \beta_1 = \frac{1}{2}(1+\cos \Psi) \]

- Can be used as filter function, too -
D) Jekeli Function: spherical equivalent of a Gauss function
(Jekeli C, OSU327, 1981):

\[
W(\psi) = \frac{b \exp\left[-b \cdot (1 - \cos \psi)\right]}{2\pi \left[1 - \exp(-2b)\right]} \quad \text{where} \quad b = \frac{\ln(2)}{1 - \cos(s/R)}
\]

(It is s= full width (arc length on earth sphere) of half value and R = earth radius or \(\Psi = s/R\))

Expansion of \(W(\psi)\) into Legendre polynomials:

\[
W_n = \int_{\psi=0}^{\pi} W(\psi) P_n(\cos \psi) \sin \psi d\psi
\]

Recursion formulae:

\[
W_{\sigma+1} = -\frac{2n+1}{b} W_{\sigma} + W_{\sigma-1} \quad \text{where} \quad W_0 = \frac{1}{2\pi} \quad \text{and} \quad W_1 = \frac{1}{2\pi} \left[1 + \exp(-2b) - \frac{1}{b}\right]
\]

- Can be used as filter function, too; compare Wahr paper -
Convenient test functions for load estimates

Gauss function $W(\psi)$, $\psi_0 = 10^\circ$

Smooth factors of Gauss function ($\psi_0 = 10^\circ$)

end
Gauss function $W(\Psi)$, $\Psi_0 = 10^\circ$

Smooth factors of Gauss function ($\Psi_0 = 10^\circ$), not normalized by 2x

Smooth factors of Gauss ($W_n$) and Pellinen ($\beta_n$) functions ($\Psi_0 = 10^\circ$)

$\beta_n$ and $W_n$
gravity und oceanography:

geoid plus altimetry =
dynamic ocean topography =
surface ocean circulation

temporal gravity variation
and sea level change:
thermal expansion versus mass increase

temporal gravity variation =
bottom pressure variation =
deep ocean circulation
from surface circulation to ocean velocity at depth
by measuring temperature and salinity profiles
(or vertical changes of pressure)

\[
\begin{align*}
    u &= -\frac{1}{f_\rho \frac{\partial}{\partial y}} \int_{\text{depth}}^{0} g(\varphi, z) \rho(z) \, dz - \frac{g}{f} \frac{\partial H}{\partial y} \\
    v &= -\frac{1}{f_\rho \frac{\partial}{\partial x}} \int_{\text{depth}}^{0} g(\varphi, z) \rho(z) \, dz + \frac{g}{f} \frac{\partial H}{\partial x}
\end{align*}
\]
Lecture: Sea Level

Maik Thomas

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

Outline

I. Introduction
II. Impacts of sea level changes
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V. Causes of sea level variations:
   i) Loading
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VII. Summary / Synthesis
What is sea level rise?

Wikipedia’s definitions

- **Sea level**: the level of the ocean’s surface
- **Mean sea level (MSL)**:
  - average height of the ocean’s surface
  - "still water level" - level of the sea with motions averaged out
  - measured in respect to land
    - $\Delta$MSL results from a real change in sea level or from a change in the height of the land on which the tide gauge operates
    - coincides with the geoid surface in a state of rest or absence of external forces
- **Local mean sea level (LMSL)**:
  - time-averaged height of the sea with respect to a land benchmark
- **Relative sea level (RSL)**:
  - distance between sea-surface and local moving datum
- **Eustatic SL changes** (as opposed to local change) result from changes in the volume of seawater or net changes in the volume of ocean basins
Relative & eustatic sea level

Sea level – an integral of changes in the Earth’s heat budget

GMSL change

- change in shape of ocean basins
- change in volume of ocean water

- change of water density
- change of ocean mass

- atmospheric warming
- precipitation & evaporation
- continental discharge
- continental ice mass balance

constraint for climate model simulations

important components of the global hydrological cycle
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The Earth at night
Some numbers ...

- MSL rise of 18 cm during the 20th century
- 1978 - 2000: 1565 km² of intertidal areas converted to open water
- by 2050 additional 1300 km² land loss

- total sea level rise (1990 - 2100) acc. to IPCC AR4:
  - 18 – 59 cm (without increasing ice discharge)
  - 18 – 76 cm (with increasing ice discharge)
  - sea level rise > 1 m also "physically plausible"

- subsidence of ...
  - megacities
    - subsidence of 2-5 m during the last century
  - coasts including river deltas
    - 9 Asian vulnerable deltas with a population of 250,000,000 people
  - threat to quality of and entry to drinking water

Potential impact of a 5 m sea level rise ...

... in Florida and Southeast Asia
Subsiding coastal megacities

maximum subsidence during the 20th century (Nicholls, 1995, Geojournal)

Which cities will flood when?

years

m

1000 20

300 4

200 3
Threatened coastal areas to 40 cm SLR by the 2080s (Nicholls et al., 1999)

Impacts of SLR
Sea level rise – response strategies

planned retreat

accommodation

protection

Options for adaptation

<table>
<thead>
<tr>
<th>NATURAL SYSTEM EFFECT</th>
<th>POSSIBLE ADAPTATION RESPONSES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Inundation, flood and storm damage</td>
<td>a. Surge Dikes/surge barriers [P],</td>
</tr>
<tr>
<td></td>
<td>b. Backwater effect Building codes/flood wise buildings [A],</td>
</tr>
<tr>
<td></td>
<td>Land use planning/hazard delineation [A/R].</td>
</tr>
<tr>
<td>2. Wetland loss (and change)</td>
<td>Land use planning [A/R],</td>
</tr>
<tr>
<td></td>
<td>Managed realignment/forced hard defences [R],</td>
</tr>
<tr>
<td></td>
<td>Nourishment/sediment management [P].</td>
</tr>
<tr>
<td>3. Erosion (of soft morphology)</td>
<td>Coast defences [P],</td>
</tr>
<tr>
<td></td>
<td>Nourishment [P],</td>
</tr>
<tr>
<td></td>
<td>Building setbacks [R].</td>
</tr>
<tr>
<td>4. Saltwater intrusion</td>
<td>a. Surface Waters Saltwater intrusion barriers [P],</td>
</tr>
<tr>
<td></td>
<td>Change water abstraction [A/R].</td>
</tr>
<tr>
<td></td>
<td>b. Ground-water Freshwater injection [P],</td>
</tr>
<tr>
<td></td>
<td>Change water abstraction [A/R].</td>
</tr>
<tr>
<td>5. Rising water tables/impeded</td>
<td>Upgrade drainage systems [P],</td>
</tr>
<tr>
<td>drainage</td>
<td>Polders [P],</td>
</tr>
<tr>
<td></td>
<td>Change land use [A],</td>
</tr>
<tr>
<td></td>
<td>Land use planning/hazard delineation [A/R].</td>
</tr>
</tbody>
</table>

P – protection; A – accommodation; R - retreat (Nicholls, 2010)
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Sea level observations

- GPS buoys
- Radar altimetry
- Tide gauges
- Gravity field recovery
- Argo floats

Courtesy: NASA

www.argo.ucsd.edu
Long-term records of relative sea level change

Distribution of PSMSL tide-gauge stations

- only a few tide gauges extend back to early 19th century
- poor coverage, both geographically and temporally
- complimentary proxy records needed!

Established before 1950  Established after 1950

Processes causing relative sea level change

- Stockholm, Sweden: Subsidence due to land uplift
- Nezugaseki, Japan: Discontinuity due to earthquake (1964)
- Bangkok, Thailand: Rise due to ground water extraction
- Manila, Philippines: Rise due to high river discharges
- Honolulu, Hawaii: “Far field” signal, no significant tectonic signal; secular trend ca. 1.5 mm/year
Sea surface observations

- determination of dynamic topography
- derivation of surface currents with geostrophic method

Sea level trend(s)

Radar altimetry since (1985) 1992

- mean global trend (60 S – 60 N): ca. 3 mm/year
- depending on period, satellites and correction models; temporal coverage still below decades

Tide gauges for up to 150 years

- pointwise measurements, offshore, heterogeneous distribution, difficult vertical control

TOPEX/Poseidon (1995)
Sea level trend(s)

TOPEX
Jason
60d-smoothing
trend: 3.2 ± 0.4 mm/yr

ΔMSL [mm]
seasonal & 60d-variations removed

Modern sea level change estimates

GMSL change since 1870 from three different studies (red, blue, black)

(IPCC AR4, 2007, Working Group 1, p. 410)
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Sea level indicators

- types:
  - sedimentary (e.g., beachrock)
  - erosional (e.g., notches)
  - ecological (e.g., accretionary bioherms constructed by coralline algae)
  - microfossils (e.g., diatoms, testate amoebae, foraminifera)
  - archaeological (e.g., fish tanks)
  - ...

- refer mostly to ocean volume
- refer, in general, to some level within the tidal range, not to MSL
- often just indicative, i.e., giving a limiting value only
- often affected by age uncertainties

| combination of geological, biological, and archaeological proxy indicators to minimize error bars |

Global sea level reconstructions during the last 500 Ma

Over most of geologic history long-term average sea level has been significantly higher than today!
Sea levels were ≈ 265 m higher than today 100 million years ago.

Main mechanisms:

1. **Sea level falls when glaciers build up on land.**
2. **Sea level falls when continental land masses are concentrated together:**
   - Collisions and mountain building squeeze Earth’s landmasses together
   - Less continental shelf area affecting the average depth of the oceans
   - Ocean basins tend to be more mature and deeper
3. **Sea level rises when new ocean basins are opening:**
   - New ocean basins generally form at only 2,500 m depth, while mature ocean basins tend to deepen to ≈ 6,000 m after 100 million years
   - Overall average depth of the ocean is lower when new oceans are forming (e.g., during the Cretaceous when the Atlantic Ocean was opening up)

Sea levels were ≈ 265 m higher than today 100 million years ago.
Global sea level reconstructions during the last 500 Ma

Sea level reconstructions (800 ky BP - present)

(DWR, 2006)
Post-glacial sea level rise

GMSL ≈ 120 m lower during Last Glacial Maximum

Post-glacial sea level rise

GMSL ≈ 120 m lower during Last Glacial Maximum

Holocene sea level

Sea Level Change (m)

Thousands of Years Ago

Australia  Jamaica  Senegal  Malacca Straits

Post-glacial sea level rise

World Sea Level at Last Ice Age, 21,000 years ago
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Short-term and periodic SL changes

<table>
<thead>
<tr>
<th>periodic SL changes</th>
<th>12h ... 24h</th>
<th>0.2m ... 10m</th>
</tr>
</thead>
<tbody>
<tr>
<td>diurnal &amp; semidiurnal tides</td>
<td></td>
<td></td>
</tr>
<tr>
<td>long-period tides</td>
<td></td>
<td></td>
</tr>
<tr>
<td>rotational variations (CW, ...)</td>
<td>14 months</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>meteorological &amp; oceanographic fluctuations</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>atmospheric pressure</td>
<td>hours ... months</td>
<td>up to 1.3m</td>
</tr>
<tr>
<td>winds (storm surges)</td>
<td>1d ... 5d</td>
<td>up to 5.0m</td>
</tr>
<tr>
<td>precipitation, evaporation</td>
<td>days ... weeks</td>
<td></td>
</tr>
<tr>
<td>sea surface topography</td>
<td>days ... weeks</td>
<td>up to 1.0m</td>
</tr>
<tr>
<td>ENSO</td>
<td>6 months every 5-10yr</td>
<td>up to 0.6m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>seasonal variations</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>river runoff/floods</td>
<td>2 months</td>
<td>1.0m</td>
</tr>
<tr>
<td>water density changes (T,S)</td>
<td>6 months</td>
<td>0.2m</td>
</tr>
</tbody>
</table>

| seiches                                     | minutes ... hours | up to 2.0m |
|                                             |                 |              |
| earthquakes                                 |                 |              |
| Tsunamis                                    | hours           | up to 10m    |
| abrupt change in land level                 | minutes         | up to 10m    |
### Long-term causes of SL change

<table>
<thead>
<tr>
<th>long-term causes</th>
<th>range of effect</th>
<th>vertical effect</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>change in volume of ocean basins</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tectonics / sea floor spreading</td>
<td>eustatic</td>
<td>0.01mm/yr</td>
</tr>
<tr>
<td>marine sedimentation</td>
<td>eustatic</td>
<td>&lt;0.01mm/yr</td>
</tr>
<tr>
<td><strong>change in ocean mass</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>melting / accumulation of cont. ice</td>
<td>eustatic</td>
<td>10mm/yr</td>
</tr>
<tr>
<td><strong>climate changes during the 20th century</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Antarctica (increasing precipitation)</td>
<td>eustatic</td>
<td>-0.2 ... 0.0mm/yr</td>
</tr>
<tr>
<td>Greenland (precipitation &amp; runoff)</td>
<td>eustatic</td>
<td>0.0 ... 0.1mm/yr</td>
</tr>
<tr>
<td><strong>long-term adjustment to the end of the last ice age</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Greenland &amp; Antarctica (20th cent.)</td>
<td>eustatic</td>
<td>0.0 ... 0.5mm/yr</td>
</tr>
<tr>
<td>water release from Earth’s interior</td>
<td>eustatic</td>
<td></td>
</tr>
<tr>
<td>release/accum. of cont. reservoirs</td>
<td>eustatic</td>
<td></td>
</tr>
<tr>
<td><strong>uplift or subsidence of Earth’s surface (isostasy)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>thermal-isostasy (interior T/ρ changes)</td>
<td>local</td>
<td></td>
</tr>
<tr>
<td>glacio-isostasy (loading of ice)</td>
<td>local</td>
<td>10mm/yr</td>
</tr>
<tr>
<td>hydro-isostasy (loading of water)</td>
<td>local</td>
<td></td>
</tr>
<tr>
<td>volcano-isostasy (magentic extrusions)</td>
<td>local</td>
<td></td>
</tr>
<tr>
<td>sediment-isostasy (deposition/erosion)</td>
<td>local</td>
<td>&lt;4mm/yr</td>
</tr>
<tr>
<td><strong>vertical effect range of effect</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>long-term causes</td>
<td>range of effect</td>
<td>vertical effect</td>
</tr>
<tr>
<td>teconic uplift / subsidence</td>
<td></td>
<td></td>
</tr>
<tr>
<td>vertical and horizontal motions of crust (in response to fault motions)</td>
<td>local</td>
<td>1-3mm/yr</td>
</tr>
<tr>
<td>sediment compaction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>sediment compression (e.g., in deltas)</td>
<td>local</td>
<td></td>
</tr>
<tr>
<td>loss of interstitial fluids (withdrawal of groundwater or oil)</td>
<td>local</td>
<td>&lt;5mm/yr</td>
</tr>
<tr>
<td>earthquake-induced vibration</td>
<td>local</td>
<td></td>
</tr>
<tr>
<td>departure from geoid</td>
<td></td>
<td></td>
</tr>
<tr>
<td>shifts in hydrosphere, aesthenosphere, core-mantle interface</td>
<td>local</td>
<td></td>
</tr>
<tr>
<td>shifts in Earth’s rotation, axis of spin, precession of equinox</td>
<td>eustatic</td>
<td></td>
</tr>
<tr>
<td>external gravitational changes</td>
<td>eustatic</td>
<td></td>
</tr>
<tr>
<td>evaporation and precipitation (if due to a long-term pattern)</td>
<td>local</td>
<td></td>
</tr>
</tbody>
</table>
What causes the sea level to change?

Atmospheric pressure variations
Steric changes
Mass induced changes
Glacial-Isostatic Adjustment
Other processes (e.g. seismicity)

(IPCC, 2007)
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Glacial isostatic adjustment (GIA)

Loading (past, 21 ka BP)
- Subsidence
- Ice sheet
- Lithosphere
- Viscous flow
- Mantle

Adjustment (present-day)
- Uplift
- Increase of gravitational potential
- Lithosphere
- Mantle

Uplift rate: up to 30 mm/a

Spatial dimension: 100s to 1000s km

(V. Klemann, GFZ)
Prediction of GIA induced sea level variations

3-dim. visco-elastic model simulation with constraints from GPS data

(Hagedoorn, 2005)

GIA correction at tide-gauge stations

Observation at tide gauges
Prediction due to GIA
GIA-reduced relative sea-level change observation

(Hagedoorn, 2005, updated)
Land movement at tide-gauges ...

... affects sea level measurements
- land movement rates at tide gauges with co-located GPS stations, e.g., from the TIGA Tide Gauge Benchmark Monitoring Pilot Project
- separation of climate related sea level change and apparent trends

Anthropogenic factors, e.g., groundwater and hydrocarbon extraction, increasing in the 20th century!

vertical trends: GPS & GIA
Comparison of GIA & GPS vertical rates

Good agreement in areas where GIA is dominant (e.g. Churchill PSML 970/141)

Large discrepancies in areas where land movement is mainly caused by anthropogenic effects or related to other geological processes (e.g. Galveston PSML 940/007(8))

GPS-GIA < 0

GPS-GIA > 0

Figure 2: 1928 photograph of a new post-quake hill 0.5 miles north of the Goose Creek oil field. The rise of the surface is approximately 15 inches. From Gelinsky et al., 1999.

Land movement: anthropogenic causes

Subsidence at Galveston tide gauge caused by groundwater extraction related to oil drilling
(Source: Houston Geological Society)

Figure 1. Measured subsidence between 1918 and 2012 around Goose Creek oil field. Lines of equal subsidence (in inches) are shown in gray (smaller) or a yellow period, in black (nearer). Modified from Tolokonov et al., 1998.

(N. Schön, 2010)
Combining GPS, tide gauges and radar altimetry for reconstruction of past sea level changes and variability

Sea level reconstructions

Original data (2.4 mm/yr)
Reconstruction using GPS corrections (1.7 mm/yr)
Reconstruction using GIA corrections (1.4 mm/yr)

(a) TOPEX sea level anomalies 12/2001
(b) Reconstruction 12/2001 using 18 GPS corrected gauges (N. Schön, 2010)

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Global hydrological cycle

Variation of total ocean mass

- total ocean mass from GRACE & ECMWF+HDM+OMCT
- cryospheric mass fluxes neglected
- linear trends removed

(Dobslaw & Thomas, 2007)
Precipitation – evaporation [mm/month]

Mass anomalies are related to recent precipitation events:
-20 -15 -10 -5 0 5 10 15 20

Terrestrial water storage anomalies

Monthly mean mass anomalies caused by continental hydrological processes as simulated with LSDM

Mass anomalies are related to recent precipitation events:
-5 0 5
Terrestrial water storage anomalies

Mean annual amplitudes of hydrological mass anomalies as observed by GRACE (2002-2009)

• currently: observation of hydrological mass redistributions
• future: removal of hydrological signals to assess secondary effects?

Mean annual amplitudes of terrestrial water storage anomalies as simulated with LSDM (2002-2007, smoothed)

Mean annual amplitudes of continental mass anomalies as observed by GRACE (2002-2009)

(Dill, GFZ)
Terrestrial water storage anomalies

Phase of mean annual
hydrological mass signal
as simulated with LSDM

Phase of mean annual signal
as observed by GRACE

Terrestrial water storage anomalies

Phase of mean annual
hydrological mass signal
as simulated with LSDM

Phase of mean annual signal
as observed by GRACE

(Dill, GFZ)
Terrestrial water storage anomalies

Correlation GRACE vs. Model

Phase of mean annual signal as observed by GRACE

Annual phase from LSDM (2002-2007) [month]

Annual phase from GRACE (2003-2009) [month]

Do we need GRACE to validate LSDM?

Do we need a model to validate GRACE?
Variability of continental water masses

WGHM model

GRACE model

correlation: > 90%

equivalent water thickness

WGHM (unconstrained)

WGHM (GRACE calibrated)

(Werth et al., 2009)

Mass anomalies in the Arctic ocean due to continental freshwater fluxes

• dominant mass signal in July up to 2hPa
• probably detectable by GRACE

(Dobslaw & Thomas, 2007)
Separation of mass change components

Ice-mass change

- Sea-level contribution

GIA

- Earth’s structure


Greenland ice mass change from 6yr GRACE

Spatial geoid height change

- Ice mass change

ca. 0.55mm contribution to global sea level change

GFZ RL04, August 2002 - August 2009, GIA corrected

(Sasgen et al., GFZ)
Greenland ice mass change from 6yr GRACE

Spatial geoid height change

Trend [mm/year]

Ice mass change

ca. 0.55mm contribution to global sea level change

Boundary conditions for climate projections?

Forward modelling of glaciation

Initialization phase (1958)

Observed heights of the ice sheet (Bamber et. al., 2001)

Height of the ice sheet after 250 kyr of palaeoclimatic simulation

<table>
<thead>
<tr>
<th></th>
<th>Observed</th>
<th>Simulated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume ($10^6$ km$^3$)</td>
<td>2.9323</td>
<td>3.0251</td>
</tr>
<tr>
<td>Area ($10^6$ km$^2$)</td>
<td>1.6681</td>
<td>1.7969</td>
</tr>
<tr>
<td>Max. elevation (km)</td>
<td>3.2741</td>
<td>3.2826</td>
</tr>
</tbody>
</table>
Ocean mass variations

Annual changes in ocean mass from GRACE, GPS & ocean modelling

Mean sea level variations

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From sea surface heights to heat transports

**total sea surface height anomalies**

- steric anomalies
- and
- mass anomalies

transformation to barotropic current anomalies associated with variations in oceanic heat transports

Temperature & salinity observations

**ARGO float**

- drifter with GNSS based positioning
- measurement of temperature & salinity up to $\approx 2000$ m depth

Distribution of ARGO floats (02/2011)

3214 floats

www.poseidon.hcmr.gr
Global mean surface temperature

(Chambers et al., 2004, GRL)

(IPCC, 2001)
Global mean surface temperature

(IPCC, 2001)

Change of surface temperatures

(NOAA)
Mean surface heat fluxes

annual mean of net surface heat exchange [W/m²] from ERA-40 re-analyses
(Kallberg et al. 2005)

Oceanic heat content

heat content in \(10^{22} \text{ J}\)

- 0 – 300m (yearly)
- 0 – 700m (yearly)
- 0 – 3000m (pentadal)

- Heat required to melt mountain glaciers at estimated maximum melting rate (Houghton et al., 2001)
- Heat required to melt northern hemisphere sea-ice (Parkinson et al., 1999)
- Heat required to melt Arctic perennial sea-ice volume (Rothrock et al., 1999)

ocean accounts for \(\approx 84\%\) of total heat storage in the Earth system

estimates of Earth’s heat balance components \(10^{22} \text{ J}\)
(1955–1998) (Levitus et al., 2005, GRL)
Thermosteric sea level anomalies [mm]

(Antonov et al., 2005, GRL)

Total vs. steric sea level variations ("trends" 2004 - 2007)

(Cazenave et al., 2008)
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Climate projections (IPCC, 2007)

change of near-surface temperatures

- Global environmental sustainability
- Rapid economic growth (homogeneous world)
- Regionally oriented economic development (heterogeneous world)
Sea level projections (IPCC, 2001):

- **Observations (1900 – 2001)** (Church & White, 2006)
- **Projections (1990 - 2100)** (IPCC, 2001)
- **IPCC (2001): scenario for 1990 - 2100**
- **Satellite altimeter**
Sea level rise: predictions for 2100

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Contributions to sea level variations

- Thermal expansion (0 – 700m)
- Thermal expansion (deep ocean)
- Antarctica & Greenland
- Glaciers & ice caps
- Terrestrial storage

Domingues et al., 2008, Nature

Contributions to sea level trends (IPCC, 2007)

- Thermal expansion
- Glaciers and ice caps
- Greenland
- Antarctica
- Sum
- Observations
- Difference (Obs-Sum)

1961 - 2003
1993 - 2003
Contributions to sea level trends (IPCC, 2007)

<table>
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<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal Expansion</td>
<td>0.42 ± 0.12</td>
<td>1.6 ± 0.5</td>
</tr>
<tr>
<td>Glaciers and Ice Caps</td>
<td>0.50 ± 0.18</td>
<td>0.77 ± 0.22</td>
</tr>
<tr>
<td>Greenland Ice Sheet</td>
<td>0.05 ± 0.12</td>
<td>0.21 ± 0.07</td>
</tr>
<tr>
<td>Antarctic Ice Sheet</td>
<td>0.14 ± 0.41</td>
<td>0.21 ± 0.35</td>
</tr>
<tr>
<td>Sum</td>
<td>1.1 ± 0.5</td>
<td>2.8 ± 0.7</td>
</tr>
<tr>
<td>Observed</td>
<td>1.8 ± 0.5</td>
<td>3.1 ± 0.7</td>
</tr>
<tr>
<td>Difference (Observed – Sum)</td>
<td>0.7 ± 0.7</td>
<td>0.3 ± 1.0</td>
</tr>
</tbody>
</table>

Future sea level rise

“forest dieback” („Waldsterben“)

Sea-level Expert: It’s Not Rising!
Vielen Dank!
Summer School “Global Water Cycle”
12.-16. September 2011, Mayschoss
DFG Priority Programme SPP1257:
Mass transport and mass distribution in the Earth’s system

Torsten Mayer-Guerr, Mail: mayer-guerr@tugraz.at
Frank Flechtner, Mail: flechtne@gfz-potsdam.de

Practical: Spherical Harmonics Synthesis

Purpose of the practical: The level-2 GRACE products are generally given as gravitatio-
nal potential in terms of spherical harmonics. In this practical, different filtered gravity field
functionals (e.g. total water storage, gravity disturbances) should be computed from these
products as gridded data on the Earth’s surface.

Directories: The required data sets can be found on the USB stick in the directory practi-
cals/data/pract1_sphericalHarmonics. The provided matlab functions are located under practi-
cals/functions/pract1_sphericalHarmonics.

Exercise 1: Spherical Harmonics Synthesis

- Load the potential coefficient files ITG-Grace2010_2008-03.gfc and ITG-
  Grace2010_2008-09.gfc using the function readPotentialCoefficients.

- Compute the difference between the coefficients of the monthly solutions.

- Filter the result with a gaussian filter (R = 500 km) using the filter coefficients computed
  by the function filterCoefficientsGauss.

- Compute gravity disturbance (in spherical approximation) on a $1^\circ \times 1^\circ$ geographical grid:

\[
\delta g(\lambda, \theta) = \frac{GM}{R^2} \sum_{n=0}^{\infty} (n+1) \sum_{m=0}^{n} \tilde{P}_{nm}(\cos \theta) \left( \tilde{C}_{nm} \cos(m\lambda) + \tilde{S}_{nm} \sin(m\lambda) \right)
\]  

(1)

The Legendre functions $\tilde{P}_{nm}$ can be computed by the function legendreFunctions.

- Visualize the result with showGrid.
Exercise 2 (optional): Spherical Harmonics Synthesis (different functionals)

- Transfer the computation of gravity disturbances into a function as defined at the end of this paper.
- Implement a function to compute geoid variations (in spherical approximation)

\[
\Delta N(\lambda, \theta) = R \sum_{n=0}^{\infty} \sum_{m=0}^{n} \bar{P}_{nm}(\cos \theta) \left( \Delta \bar{C}_{nm} \cos(m\lambda) + \Delta \bar{S}_{nm} \sin(m\lambda) \right). \tag{2}
\]

- Implement a function to compute total water storage change (common approximation)

\[
\Delta TWS(\lambda, \theta) = \frac{\rho_e R}{3} \sum_{n=0}^{\infty} \frac{2n + 1}{1 + k'_n} \sum_{m=0}^{n} \bar{P}_{nm}(\cos \theta) \left( \Delta \bar{C}_{nm} \cos(m\lambda) + \Delta \bar{S}_{nm} \sin(m\lambda) \right), \tag{3}
\]

with the mean density of the Earth \( \rho_e = 5540 \text{ kg/m}^3 \). The load love numbers \( k'_n \) are given in the file loadLove.mat.
Matlab functions:

**function 
\[\text{cnm, snm, GM, R} = \text{readPotentialCoefficients} \left( \text{filename} \right) \]
Read potential coefficients from file.

<table>
<thead>
<tr>
<th>Input</th>
<th>• \textit{filename}: file of potential coefficients in ICGEM format (*gfc).</th>
</tr>
</thead>
</table>
| Output| • \textit{cnm, snm}: matrices containing the potential coefficients. The dimensions are \((n+1) \times (n+1)\) with \(n = \text{maxDegree}\). The lower triangular matrix elements \(c_{n,m+1}^{n+1} \) and \(s_{n,m+1}^{n+1} \) contain the potential coefficients of degree \(n\) and order \(m\).  
• \textit{GM}: Earth gravity constant.  
• \textit{R}: Earth reference radius. |

**function 
\[\text{Pnm} = \text{legendreFunctions} \left( \text{theta, maxDegree} \right) \]
Calculation of all Legendre Functions (4\(\pi\)-normalized) up to given degree and order at a specific co-latitude.

| Input | • \textit{theta}: co-latitude in radians.  
• \textit{maxDegree}: maximum degree and order to compute. |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>• \textit{Pnm}: matrix containing the 4(\pi)-normalized Legendre functions. The dimension is ((n+1) \times (n+1)) with (n = \text{maxDegree}). The lower triangular matrix element (P_{n,m+1}^{n+1} ) contains the Legendre function of degree (n) and order (m).</td>
</tr>
</tbody>
</table>

**function 
\[\text{wn} = \text{filterCoefficientsGaussian} \left( \text{radius, maxDegree} \right) \]
Filter coefficients in the spectral domain of a Gaussian filter.

| Input | • \textit{radius}: half-with radius parameter in km.  
• \textit{maxDegree}: maximum degree to compute. |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>• \textit{wn}: ((n+1) \times 1) vector with (n = \text{maxDegree}). The vector element (w_{n+1}^{n+1} ) contains the filter coefficient of degree (n).</td>
</tr>
</tbody>
</table>

**function 
\[\text{showGrid} \left( \text{lambda, theta, grid, title} \right) \]
Plot gridded data on Earth's surface. To plot the coast lines, the file \textit{coast.dat} must be in the working directory.

| Input | • \textit{lambda}: \(p \times 1\) vector containing the longitudes (in radians).  
• \textit{theta}: \(q \times 1\) vector containing the co-latitudes (in radians).  
• \textit{dg}: \(q \times p\) matrix containing the gridded values.  
• \textit{title}: the title text of the figure. |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>•</td>
</tr>
</tbody>
</table>
Matlab to implement in this exercise:

<table>
<thead>
<tr>
<th>function dg = gravityDisturbance(lambda, theta, cnm, snm, GM, R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate gravity disturbances as gridded data defined by the vectors lambda and theta.</td>
</tr>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>• lambda: $p \times 1$ vector containing the longitudes (in radians).</td>
</tr>
<tr>
<td>• theta: $q \times 1$ vector containing the co-latitudes (in radians).</td>
</tr>
<tr>
<td>• cnm,snm: matrices containing the potential coefficients</td>
</tr>
<tr>
<td>• GM: Earth gravity constant.</td>
</tr>
<tr>
<td>• R: Earth reference radius.</td>
</tr>
<tr>
<td><strong>Output</strong></td>
</tr>
<tr>
<td>• dg: $q \times p$ matrix containing the gridded gravity anomalies.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>function dN = geoid(lambda, theta, cnm, snm, GM, R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate geoid changes as gridded data defined by the vectors lambda and theta in spherical approximation</td>
</tr>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>• lambda: $p \times 1$ vector containing the longitudes (in radians).</td>
</tr>
<tr>
<td>• theta: $q \times 1$ vector containing the co-latitudes (in radians).</td>
</tr>
<tr>
<td>• cnm,snm: matrices containing the potential coefficients</td>
</tr>
<tr>
<td>• GM: Earth gravity constant.</td>
</tr>
<tr>
<td>• R: Earth reference radius.</td>
</tr>
<tr>
<td><strong>Output</strong></td>
</tr>
<tr>
<td>• dN: $q \times p$ matrix containing the gridded geoid variations.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>function tws = totalWaterStorage(lambda, theta, cnm, snm, GM, R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate total water storage changes as gridded data defined by the vectors lambda and theta in spherical approximation. The file loadLove.mat must be in the working directory.</td>
</tr>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>• lambda: $p \times 1$ vector containing the longitudes (in radians).</td>
</tr>
<tr>
<td>• theta: $q \times 1$ vector containing the co-latitudes (in radians).</td>
</tr>
<tr>
<td>• cnm,snm: matrices containing the potential coefficients</td>
</tr>
<tr>
<td>• GM: Earth gravity constant.</td>
</tr>
<tr>
<td>• R: Earth reference radius.</td>
</tr>
<tr>
<td><strong>Output</strong></td>
</tr>
<tr>
<td>• tws: $q \times p$ matrix containing the total water storage.</td>
</tr>
</tbody>
</table>
Practical: Analysis Tools

Purpose of the practical: In the practical “Analysis Tools” the method of “Principal Component Analysis“ (PCA) will be applied to time series of GRACE data and to hydrological model output provided by the WaterGAP Hydrology Model (WGHM). PCA is used to extract individual dominant modes of the data variability, while simultaneously suppressing those modes connected with low variability and therefore reducing the dimension of data efficiently. The given time-space data field (e.g. monthly data given on a geographical grid) is separated into spatial structures called empirical orthogonal functions (EOF) and their amplitudes in time, called principle components (PCs).

Additional material: Lecture Notes “Analysis Tools” by Jürgen Kusche, Annette Eicker and Ehsan Forootan

Directories: The Data sets needed for this practical can be found on the USB stick in the directories practicals/data/generic and practicals/data/pract2_analysisTools. The provided matlab functions can be found in the directory practicals/functions/pract2_analysisTools.

Exercise 1: Calculation and visualization of EOFs and PCs

- Load the files wghm_filtered.mat and grace_filtered.mat. Each file contains a monthly time series of filtered gridded values (Gauss filter with 400km radius, years 2005-2008) on a 1° × 1° geographical grid stored in the data matrix Y. The data sets are centered, i.e. they have zero mean. The gridded values of one month are sorted into one column of the matrix. Thus Y has the dimensions n × p with n = number of grid points and p = number of points in time. Furthermore, the files each contain two vectors with the dimensions n×1, containing the longitude values of the grid points (lon) and the latitude values of the grid points (lat).
  - Calculate the spatial patterns (EOFs) from the WGHM data set using the function calculateEOF. These patterns serve as basis functions for further calculations.
  - Visualize the spatial patterns for the first three modes using the function showEOF.
  - Calculate the temporal evolution (PCs) of the above calculated basis functions both from the GRACE and the WGHM data using the function calculatePC.
  - Visualize the temporal evolution for the first three modes, both for GRACE and WGHM using the function showPC.
• Compare the results for GRACE and WGHM.

Exercise 2: Understanding compression properties of PCA

• Visualize the eigenvalues calculated in Exercise 1. You can use the function `showEigenvalues` for an easy visualization.

• How many modes are needed to reconstruct 80% and 95% of the WGHM variability? Calculate the fraction of the signal variability reconstructed by $\bar{p}$ modes according to

$$ var_{\bar{p}} = \frac{\sum_{j=1}^{\bar{p}} \lambda_j}{\Delta^2} $$

with the total variance

$$ \Delta^2 = \sum_{j=1}^{P} \lambda_j. $$

(1)

• From a given matrix $E$ containing the EOFs in its columns and a matrix $D$ containing the PCs in its rows, the signal matrix $Y$ can be reconstructed by

$$ Y = ED. $$

(2)

Use the calculated PCA from the global WGHM time series to reconstruct 80% and 95% of the variability.

• Plot the reconstructed signal (80%, 95%) and the original signal and the difference between both using the function `showData` for one arbitrary month (e.g. 2005-05).

Exercise 3 (optional): Understanding domain dependence of PCA

• Use the global WGHM time series.

• Cut out the data in the Amazon region and in the Orinoco region. You can use the function `inpolygon` provided by matlab. The boundary polygons for the two regions are provided by the files `amazon.mat` and `orinoco.mat`. Each file contains a matrix with two columns, the first column consisting of the longitude values of the polygon points, the second column containing the latitude values.

• Visualize only the first EOF and PC. Here you can use the function `showEOFlocal`.

• How many modes would be necessary to reconstruct 95% of the signal?

• Compare the two regional results and the global results (from Exercise 1 and 2) and discuss.
Matlab functions:

function \[eigenvalues, E\] = calculateEOF(Y)

Calculation of EOFs from a given data matrix \(Y\). The function calculates the \(p\) eigenvectors (=EOFs) corresponding to the \(p\) non-zero eigenvalues of the covariance matrix \(C = YY^T\).

To reduce the computation effort, the eigenvalue problem is in a first step solved for the smaller matrix \(C' = Y^T Y\), which has the same eigenvalues. The eigenvectors of the larger matrix can then be calculated from the eigenvectors of the smaller matrix. The eigenvalues are stored in the vector \textbf{eigenvalues} sorted according to their magnitude, the eigenvectors are returned in the matrix \(E\).

Input

- \(Y\): matrix containing the time series of gridded data sets. The dimension is \(n \times p\) with \(n\) = number of grid points and \(p\) = number of points in time.

Output

- \textbf{eigenvalues}: \(p \times 1\) vector containing the \(p\) non-zero eigenvalues of the covariance matrix \(C\), sorted according to decreasing size
- \(E\): \(n \times p\) matrix containing in its columns the eigenvectors (EOFs) of the covariance matrix \(C\). EOFs are sorted according to size of corresponding eigenvalue.

function \[D\] = calculatePC(E, Y)

Calculation of principle components (PCs) by projecting the original data onto the basis of the EOFs. The PCs are stored in the rows of the matrix \(D\)

Input

- \(Y\): matrix containing the time series of gridded data sets. The dimension is \(n \times p\) with \(n\) = number of grid points and \(p\) = number of points in time
- \(E\): \(n \times p\) matrix containing in its columns the eigenvectors (EOFs) of the covariance matrix \(C\). EOFs are sorted according to size of corresponding eigenvalues.

Output

- \(D\): \(p \times p\) matrix containing the principle components in its rows.

Functions for visualization:

function showEigenvalues (Eigenvalues)

Visualization of the evolution of the eigenvalues

Input

- \textbf{Eigenvalues}: vector of eigenvalues

Output

- Plot of the eigenvalues
### function showEOF (EOF, longitude, latitude, i, titleString)

Visualization of the spatial pattern of one specific EOF with the index i.

**Input**
- **EOF**: $n \times 1$ vector containing the gridded values of the one EOF to be plotted. E.g. one column of the matrix $E$.
- **longitude**: $n \times 1$ vector containing the longitude values of the grid points
- **latitude**: $n \times 1$ vector containing the latitude values of the grid points
- **i**: index of the EOF to be visualized (EOFs are sorted according to size of eigenvalue), needed for title of the plot
- **titleString**: a string describing the data source (e.g. “WGHM”), used for the title of the plot.

**Output**
- Plot of the spatial pattern

### function showEOFlocal (EOF, longitude, latitude, border, i, titleString)

Visualization of the spatial pattern of one specific EOF with the index i on a spatial domain limited by the polygon **border**. The dimension $n$ refers to the number of grid points within the polygon.

**Input**
- **EOF**: $n \times 1$ vector containing the gridded values of the one EOF to be plotted. E.g. one column of the matrix $E$.
- **longitude**: $n \times 1$ vector containing the longitude values of the grid points
- **latitude**: $n \times 1$ vector containing the latitude values of the grid points
- **border**: matrix which contains the polygon around the regional area (e.g. Amazon). The first column consists of the longitude values of the polygon points, the second column contains the latitude values.
- **i**: number of the EOF to be visualized (EOFs are sorted according to size of eigenvalue), needed for title of the plot
- **titleString**: A string describing the data source (e.g. “WGHM”), used for the title of the plot.

**Output**
- Plot of the spatial pattern for a regional area

### function showPC (PC, i, titleString)

Visualization of the principle component with the index $i$.

**Input**
- **PCs**: $1 \times p$ vector containing the principle component which shall be visualized. E.g. the $i$-th row of the matrix $D$.
- **i**: number of the PC to be visualized (sorted according to the size of the eigenvalues), needed for title of the plot
- **titleString**: A string describing the data source (e.g. “WGHM” or “GRACE”), used for the title of the plot.

**Output**
- Plot of the principle component.
function showData(data, longitude, latitude)

Visualization of (global) gridded data sets. All gridded values are given in a column vector `data`. The longitude and latitude values corresponding to the grid values are given in the vectors `longitude` and `latitude`.

<table>
<thead>
<tr>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>• <code>data</code>: $n \times 1$ vector containing the data values, e.g. one column of the data matrix $Y$.</td>
</tr>
<tr>
<td>• <code>longitude</code>: $n \times 1$ vector containing the longitude values of the grid points</td>
</tr>
<tr>
<td>• <code>latitude</code>: $n \times 1$ vector containing the latitude values of the grid points</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>• 2D-plot of the gridded values</td>
</tr>
</tbody>
</table>
Practical: Analysis Tools

Annette Eicker

Summer School „Global Water Cycle“
12.-16. September 2011
Mayschoss

Download solutions to practical exercises:
ftp://skylab.itg.uni-bonn.de/summerschoolDownload/solutions/
Practical: Analysis Tools

**Goal:**
Understanding the method of Principal Component Analysis (PCA)

Application of PCA to gridded data sets of **GRACE** solutions and hydrological model output (**WGHM**), both given as equivalent water height, zero mean


---

Part II: Principle component analysis and related ideas

What do we get from PCA?

- **PC₁**: t → 62% → **EOF₁**
- **PC₂**: t → 24% → **EOF₂**
- **PCₙ**: t → <1% → **EOFₙ**
Data matrix

Time series provided as data matrix:

\[ Y = (y_1, y_2, \ldots, y_p) = \begin{pmatrix} y_{1;1} & y_{1;2} & \cdots & y_{1;p} \\ y_{2;1} & y_{2;2} & \cdots & y_{2;p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n;1} & y_{n;2} & \cdots & y_{n;p} \end{pmatrix} \]

\# points in time e.g. 48 months

\# locations e.g. 64,000 grid points

gridded data for one point in time, e.g. one monthly solution

grace_filtered.mat, wghm_filtered.mat

Eigenvalue decomposition

Calculation of the temporal covariance matrix:

\[ C = \frac{1}{p} Y Y^T = \frac{1}{p} \left( \sum_{i=1}^{p} y_{1;i}^2 \right) \hspace{1cm} \begin{pmatrix} \sum_{i=1}^{p} y_{1;i} y_{1;i} & \sum_{i=1}^{p} y_{1;i} y_{2;i} & \cdots & \sum_{i=1}^{p} y_{1;i} y_{n;i} \\ \sum_{i=1}^{p} y_{2;i} y_{1;i} & \sum_{i=1}^{p} y_{2;i} y_{2;i} & \cdots & \sum_{i=1}^{p} y_{2;i} y_{n;i} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{p} y_{n;i} y_{1;i} & \sum_{i=1}^{p} y_{n;i} y_{2;i} & \cdots & \sum_{i=1}^{p} y_{n;i} y_{n;i} \end{pmatrix}_{n \times n} \]

Eigenvalue decomposition of the covariance matrix:

\[ C = \Lambda E E^T = \left( e_1, e_2, \ldots, e_n \right) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \left( e_1^T, e_2^T, \ldots, e_n^T \right) \]
**Eigenvalue decomposition**

Calculation of the spatial covariance matrix:

\[
C' = \frac{1}{n} YY^T = \frac{1}{n} \begin{pmatrix}
\sum_{j=1}^{n} y_{j,1}^2 & \sum_{j=1}^{n} y_{j,1} y_{j,2} & \cdots & \sum_{j=1}^{n} y_{j,1} y_{j,p} \\
\sum_{j=1}^{n} y_{j,2} y_{j,1} & \sum_{j=1}^{n} y_{j,2}^2 & \cdots & \sum_{j=1}^{n} y_{j,2} y_{j,p} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{j=1}^{n} y_{j,p} y_{j,1} & \sum_{j=1}^{n} y_{j,p} y_{j,2} & \cdots & \sum_{j=1}^{n} y_{j,p}^2 \\
\end{pmatrix}
\]

The matrices \( C \) and \( C' \) have the same \( p \) non-zero eigenvalues.

The \( p \) corresponding eigenvectors \( e_i \) of \( C \) can be calculated from eigenvectors of \( C' \).

Matrix containing the EOFs: 

\[
E = \begin{pmatrix} e_1 & e_2 & \cdots & e_p \end{pmatrix}_{n \times p}
\]

Sorted according to size of eigenvalues:

\( \lambda_1 > \lambda_2 > \ldots > \lambda_p \)

**Principle components**

Eigenvectors (= EOFs) constitute a new orthogonal basis. PCs are calculated by projecting the data onto the new basis:

\[
D = E^T Y
\]

**Signal reconstruction:**

\[
Y = ED
\]
Signal reconstruction

Signal reconstruction:

\[ Y = ED \]

Compression: using only the first \( p \) „major“ EOFs and PCs for the reconstruction as they contain most of the variability

Signal variability using \( p \) EOFs:

\[
\text{var} = \frac{\sum_{j=1}^{p} \lambda_j}{\Delta^2} \quad \text{with} \quad \Delta^2 = \sum_{j=1}^{p} \lambda_j
\]

(total variance)

Practical 2

Exercise 1: Calculation and visualization of EOFs and PCs
- calculate EOFs from WGHM
- calculate corresponding PCs from WGHM and GRACE

Exercise 2: Compression properties
- reconstruct 80% or 95% of the data

Exercise 3 (optional): Domain dependence
- repeat Exercise 1 & 2 for two individual river basins
Results

Exercise 1
Calculation and visualization of EOFs and PCs
Exercise 2

Understanding compression properties of PCA
Signal reconstruction 2005-05

Original data: 48 monthly grids
8 EOFs needed to reconstruct 95%
3 EOFs needed to reconstruct 80%

Differences

Exercise 3
Understanding domain dependence of PCA
PCA WGHM 2005 - 2008

3 EOFs needed to reconstruct 95%}

1 EOFs needed to reconstruct 95%
Practical: Ice and Loading

**Purpose of the practical:** In the practical 'Ice and Loading' the elastic response of the Earth and the ocean to a changing ice load will be investigated. For this purpose, ICESat-derived ice height changes over Greenland are provide and will be used as input for solving the 'sea-level equation' iteratively. The inferred present-day changes of the Earth’s gravity field will be combined with GIA-induced viscous effects. This combined result and its comparison to the GRACE-derived geoid changes form the basis for concluding discussions.

**Additional material:** Lecture Notes 'Ice' by Reinhard Dietrich and Lecture Notes 'Loading' by Volker Klemann. Additional information can be found within this document.

**Directories:** The required data sets can be found on the USB stick in the directory `practicals/data/pract4_iceLoading`. The example scripts are located under `practicals/functions/pract4_iceLoading` and should be started from this directory. Utilised matlab functions can be found in the subdirectory `mtools`. 
1 Exercise 1: Solving the sea-level equation for the elastic case

We want to calculate the adjustment of the sea level due to a loading process. Input is a load distribution, which is the spatial distribution of an ice-height change. Ice-mass changes are usually represented by a change in ice thickness or in equivalent water thickness. Here, loadfile contains ice-thickness changes. As discussed during the lecture, the sea-level equation (SLE) is an integral equation and will be solved here iteratively (Sec. 1.1, p. 2). The convolution integral is presented in Sec. 1.2, p. 3.

In the matlab script sle.pt, the i/o and iteration procedure is prepared.

Here, we give a small description of the steps to work through.

1. Definition of global parameters.
2. Definition of Input files. The LLN, the topography and the ice-height changes are provided.
3. Read in of LLN and definition of the Green’s function needed for the SLE.

The output of each iteration is presented on the screen and will be stored in ../../data/pract4_iceLoading/output_data/rsl_XXX_YY.dat, here XXX denotes the maximum degree and YY the number of iterations.

In sle.pt.m, the number of iteration steps is predefined. How can this be improved?

1.1 Iterative procedure

The steps of the iteration are represented below, where * denotes the convolution of the appropriate Green’s function.

\[
(e - u)_i(\Omega) = g_{e-u} * [m_{\text{load}}(\Omega) + s_{i-1}(\Omega) \rho w \mathcal{O}(\Omega)] ,
\]

\[
s_i(\Omega) = s^\text{esl}_i + (e - u)_i(\Omega) \mathcal{O}(\Omega) ,
\]

\[
s^\text{esl}_i = -\int_{\Omega} m_{\text{load}}(\Omega) \frac{1}{\rho w A_o} - \frac{1}{A_o} \int (e - u)_i(\Omega) \mathcal{O}(\Omega) d\Omega
\]

\[
s_0 = -\int m_{\text{load}}(\Omega) \frac{1}{\rho w A_o} ,
\]

(1)
Here, \( e \) is the displacement rate of the reference equipotential, \( u \) is the vertical displacement rate, \( m_{\text{load}} \) is the distribution of the load change, \( s_i \) represents the relative sea level, \( \rho_w \) is the density of sea water and \( \mathcal{O} \) is the ocean function. \( \Omega = (\theta, \phi) \) represents the coordinates on the sphere. Furthermore, \( s_i^{\text{est}} \) is the equivalent sea level due to mass conservation and deformation and \( A_o \) is the ocean surface. Alternatively the geoid, \( N \), representing the actual averaged sea-level height, is defined as

\[
n(\Omega) = s_i^{\text{est}} + e(\Omega)
\]

For the first approximation, \( s_0^{\text{est}} \) only consists of the uniform sea-level change due to mass change of the load. Iteration steps \( i \in \{1, ..., 3\} \) should be sufficient to reach convergence.

1.2 Convolution

The convolution in (1) can be replaced by multiplication in the spectral domain:

\[
[g_\alpha * m](\Omega) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} G^e_{l,m} \cdot M_l \cdot Y_m(\Omega)
\]

Here, \( g_\alpha \) represents any Green’s function, which are expressible for the elastic response to a surface load by combinations of load Love numbers (LLN), and \( M_l \) is the spectral representation of the load, \( m \). The most prominent Green’s functions are for the potential displacement,

\[
G^e_l = \frac{3}{\bar{\rho}} \left( 1 + k_l \right) \frac{1}{2l + 1}
\]

and for the vertical displacement

\[
G^u_l = \frac{3}{\bar{\rho}} (h_l) \frac{1}{2l + 1}
\]

from which the linear combination for the case of the relative sea level follows

\[
G^{e-u}_l = G^e_l - G^u_l.
\]

In these equations, the mass of the Earth is replaced by its average density, \( \bar{\rho} \).
2 Exercise 2: Comparison of GIA-induced and present-day geoid changes with GRACE

We want to combine the displacement of an equipotential surface, \( \epsilon \), of the Earth’s gravity field due to a changing ice and ocean load, discussed in Exercise 1 (Sec. 1, p. 2), with those induced by GIA. The replacement of the geoid by the potential displacement is valid, as long as we do not consider the degree 0 component, see lecture note ‘Loading’. This combined result should be compared to the results derived from GRACE. All calculation will be performed in the spherical-harmonic domain. Just the graphical presentation of the results requires a transformation into the spatial domain.

In the matlab script `compare.pt`, the i/o and calculation procedure is prepared.

Here, we give a small description of the steps to work through.

1. Definition of global parameters.
2. Definition of Input files. The LLN and the topography are provided.
3. Read in of LLN and definition of the Green’s function needed.
5. Read in of GRACE stokes trend coefficients and conversion into geoid trend coefficients.
6. Read in of GIA equipotential surface trend coefficients.
7. Read in of the relative sea-level change from the last iteration of Exercise 1 and the ice height changes.
8. Combination of the later and transformation into spherical harmonic coefficients.
9. Perform the convolution in order to derive present-day geoid changes.
10. Combination of GIA and present-day geoid changes and comparison to GRACE results.

Again, your task will be to identify and program the convolution integral. Moreover, you should combine GIA and present-day effects in the spherical harmonic domain and compare them to the GRACE results. Therefore, plots in the space domain should be prepared.

How can the remaining differences, if there are any, be explained?
## 3 Matlab/ocative functions:

### Input

<table>
<thead>
<tr>
<th>function</th>
<th>[shc] = icgem2kff(fname [, nmax])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reads a file of spherical harmonic coefficients in ICGEM format and transfers it to a 2d-array containing $n, m, cnm, snm$. Optionally the output can be reduced to a maximum degree, $n_{\text{max}}$.</td>
<td></td>
</tr>
</tbody>
</table>

#### Input

- **fname**: Input file in ICGEM format.
- **[nmax]**: Optional maximum degree of coefficients to be read.

#### Output

- **n, m, cnm, snm**: 2d-arrays containing degree, order and the spherical potential coefficients. Dimension is $(n+1)(n+2)/2$ with $n=n_{\text{max}}$ or maximum degree read in.

<table>
<thead>
<tr>
<th>function</th>
<th>[arr, [lon, lat]] = xyz2arr(fname, lonres, latres)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reads lon, lat, value from a file and stores the values in a 2d global array.</td>
<td></td>
</tr>
</tbody>
</table>

#### Input

- **fname**: File containing a global regular grid in lon-lat convention.
- **lonres**: Longitudinal grid sampling.
- **latres**: Latitudinal grid sampling.

#### Output

- **arr**: 2d-array containing the grid values $[(90 : \text{latres} : -90) \times (0 : \text{lonres} : 360)]$.
- **[lon, lat]**: 1d arrays containing longitude $[360/\text{lonres} + 1]$ and latitude $[180/\text{latres} + 1]$.

<table>
<thead>
<tr>
<th>function</th>
<th>[n, h ,k ,l] = lln2arr(fname [, nmax])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reads load Love numbers from a file containing $n, h, k, l$ and transfers them to arrays containing degrees $n$ starting with 0 and corresponding $h, k$ and $l$.</td>
<td></td>
</tr>
</tbody>
</table>

#### Input

- **fname**: Input file containing the load Love numbers.
- **nmax**: Optional maximum degree of numbers read in.

#### Output

- **n, h, k ,l**: 1d arrays containing degree and respective load Love numbers.
## Output

<table>
<thead>
<tr>
<th>kf2icgem(fname, shc, nmax, type, name)</th>
<th>Writes a set of spherical harmonic coefficients (shc) to a file in ICGEM format.</th>
</tr>
</thead>
</table>
| **Input**                           | • **fname**: File to which the coefficients are written (in ICGEM format).  
|                                     | • **shc**: 2d-array holding $n, m, c_{nm}, s_{nm}$.  
|                                     | • **nmax**: Maximum degree of the coefficients that are written out.  
|                                     | • **type**: Header information on the product type (one string without spaces).  
|                                     | • **name**: Header information on the model name (one string without spaces).  |
| **Output**                          | ASCII file. |

<table>
<thead>
<tr>
<th>function arr2xyz(fname, arr)</th>
<th>Extracts lon, lat, value from a global 2d-array and writes them to a file.</th>
</tr>
</thead>
</table>
| **Input**                           | • **fname**: Name of the output file to which lon, lat, value are written.  
|                                     | • **arr**: 2d-array containing the grid values $[(90 : \text{latres} : -90) \times (0 : \text{lonres} : 360)]$.  |
| **Output**                          | ASCII file. |
**Analysis and Synthesis for Spherical harmonics representation**

The two principle routines which are called from sle_pt.m are plm2xyz and xyz2plm which are listed below. These together with a number of further routines are provided by Frederik Simons, MIT, which can be downloaded from http://geoweb.princeton.edu/people/simons/software.html. The respective routines used in this practical can be found in *mtools/simons*.

**function [r,lon,lat,Plm]=plm2xyz(lmcosi,degres,c11cmn,lmax,latmax,Plm)**

Inverse spherical harmonic transform.

Compute a spatial field from spherical harmonic coefficients given as \([l\ m\ \text{Ccos}\ \text{Csin}]\) (not necessarily starting from zero, but sorted), with degree resolution 'degres' [default: approximate Nyquist degree]. Using 4*pi-normalized real spherical harmonics.

**Input**

- **lmcosi**: Matrix listing l,m,cosine and sine expansion coefficients e.g. those coming out of ADDMON
- **degres**: Longitude/ latitude spacing, in degrees [default: Nyquist]
  OR
  "lat": a column vector with latitudes [degrees]
  OR
  [longitude-latitude-spacing], in degrees in combination with corner nodes in c11cmn
- **c11cmn**: Corner nodes of lon/lat grid [default: 0 90 360 -90] OR "lon": a column vector with longitudes [degrees]
- **lmax**: Maximum bandwidth expanded at a time [default: 720]
- **latmax**: Maximum linear size of the latitude grid allowed [default: Inf]
- **Plm**: The appropriate Legendre polynomials should you already have them

**Output**

- **r**: The field (matrix for a grid, vector for scattered points)
- **lon, lat**: The grid (matrix) or evaluation points (vector), in degrees
- **Plm**: The set of appropriate Legendre polynomials should you want them

**function [lmcosi,dw]=xyz2plm(fthph,L,method,lat,lon,cnd)**

Forward real spherical harmonic transform in the 4pi normalized basis.

Converts a spatially gridded field into spherical harmonics. For complete and regular spatial samplings \([0 \ 360 \ -90 \ 90]\). If regularly spaced and complete, do not specify lat,lon. If not regularly spaced, fthph, lon and lat are column vectors.

**Input**

- **fthph**: Function defined on colatitude theta and longitude phi
- **L**: Maximum degree of the expansion (Nyquist checked)
- **method** 'gl' By Gauss-Legendre integration (fast, inaccurate)
  'simpson' By Simpson integration (fast, inaccurate)
  'irr' By inversion (irregular samplings)
  'im' By inversion (fast, accurate, preferred)
  'fib' By Riemann sum on a Fibonacci grid (not done yet)
- **lat**: If not \([90,-90]\), give latitudes explicitly, in degrees
- **lon**: If not \([0,360]\), give longitudes explicitly, in degrees
- **cnd**: Eigenvalue tolerance in the irregular case
### Output
- lmcosi Matrix listing l,m,cosine and sine coefficients
- dw Eigenvalue spectrum in the irregular case

### Visualisation of fields

<table>
<thead>
<tr>
<th>plot_grid(lon, lat, arr, label [,fname])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plots xyz-field arr to screen and optionally to an eps-file.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>• lon, lat: 1d arrays containing longitude ([360/\text{lonres} + 1]) and latitude ([180/\text{latres} + 1])</td>
</tr>
<tr>
<td>• arr: 2d-array containing the grid values ([(90 : \text{latres} : -90) \times (0 : \text{lonres} : 360)]).</td>
</tr>
<tr>
<td>• label: Colorbar label.</td>
</tr>
<tr>
<td>• [fname]: Name of the eps-file.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>N/A</td>
</tr>
</tbody>
</table>

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4 Usage of matlab/octave functions

4.1 Input

octave:1> [kff]=icgem2kff("infile", nmax)
   reads in ICGEM file and transfers it to [kff]=[degree, order, cos, sin] starting with degree=0.

octave:2> [arr]=xyz2arr("infile", lonres, latres)
   reads in regular and global [lon, lat, value] list (resolution: lonres x latres) to global array.

octave:3> [degree, h, k, l]=lln2arr("infile", nmax)
   reads in [degree, h, k, l] list and writes it to 1d-arrays degree, h, k, l.

4.2 Output

octave:1> kff2icgem("outfile", kff, nmax, type, name)
   writes kff=[degree, order, cos, sin] to an ASCII file in ICGEM format starting with degree=0. Type and name provide ICGEM format header information.

octave:2> arr2xyz("outfile", arr) writes global array, arr to [lon, lat, value] list.

4.3 SH synthesis

octave:1> load -ascii ugeod.dat
   reads in a spectral array [deg order cos sin].

octave:2> [u,lon, lat, plm] = plm2xyz(ugeod, [0.351562 0.35122], [0 89.7367 360 -89.7367], 170)
   generates spatial field u, here for an equidistant grid approximating a GL-grid of 2025 x 512 grid points.

octave:3> help plm2xyz
   provides the information about the command.

4.4 SH analysis

octave:1> [kff] = xyz2plm(arr, nmax,'im')
   transforms a regular global grid arr (as generated by xyz2arr) into spherical harmonic coefficients of maximum degree nmax, which are stored in [kff]=[degree, order, cos, sin] starting with degree=0.

octave:2> help xyz2plm
   provides the information about the command.
4.5 Visualisation of fields

octave:1> plot_grid(lon, lat, z, 'zlabel', ['out.eps'])
plots xyz-field z to screen and optionally to ‘out.eps’. lon, lat, z are the 1d-longitude, -latitude arrays and the 2d-z array to be plotted, ’zlabel’ is the name of the annotation of the color bar and ’out eps’ is the the optional ps file.
Practical: Altimetry and Ocean Dynamics

Purpose of the practical:

Information about the density distribution in the global ocean as provided by in-situ observations of temperature and salinity allows to derive the geostrophic shear field. Absolute geostrophic currents might be subsequently obtained by either assuming or observing the geostrophic velocities at a certain depth level. The dynamic ocean topography as derived from the difference between sea surface and geoid provides the required information to derive such currents along the surface.

- combine altimetry and geoid height to obtain dynamic topography field
- compute surface geostrophic velocities
- compute 3D-geostrophic velocity field in the ocean from hydrography and surface dynamic topography
- compare 3D-geostrophic velocity field with numerical ocean model

Input Data:

- geoid height field, unfiltered and filtered (dgfi_alongtrack.mat)
- sea surface height (SSH) from Jason 1 for five different cycles, unfiltered and filtered, along-track and gridded (dgfi_alongtrack.mat)
- multi-year mean dynamic topography (dgfi_meandot.mat)
- WOCE hydrographic atlas: in-situ temperature and salinity (woce_climatology.mat), additional information: http://icdc.zmaw.de/woce.html
- 3D velocity field and dynamic topography field from ECCO2 ocean general circulation model (ecco2_data.mat), additional information: http://ecco2.org/

Matlab Libraries and Functions:

- CSIRO Sea Water Library (seawater_ver3_0), additional information: http://www.cmar.csiro.au/datacentre/ext_docs/seawater.htm
- Gibbs-Sea Water Oceanographic Toolbox (gsw_matlab_v3_0), additional information: http://www.teos-10.org/software.htm
Exercise 1: Dynamic Topography

- load and plot gridded sea surface heights $h$ and geoid heights $N$ for cycle 101 ($dgfi_alongtrack.mat$)
- calculate and plot DOT $\eta$ from unfiltered and filtered $h$, $N$, and compare
- derive surface geostrophic velocity field from grid spacing information and $\eta$
- optionally: project along-track data from another cycle onto the grid and compare $\eta$, $v$ with previous results

Exercise 2: Geostrophic flow field

- load multi-year DOT $\eta$ ($dgfi_meandot.mat$) and derive surface geostrophic velocities as in exercise 1
- load hydrographic climatology ($T$, $S$) ($woce_climatology.mat$)
- compute in-situ density $\sigma = \rho - 1000$, potential densities $\sigma_{0,2,4}$, neutral density $\gamma_n$ (functions: sw_dens and sw_pden, alternatively corresponding functions from the GSW Toolbox) and compare
- compute geostrophic shear $\partial v/\partial z$ from thermal wind ($-\frac{\omega}{f} k(\nabla \times \rho)$), and, optionally from “dynamic method” via geopotential height anomaly (functions: sw_gpan.m and sw_gvel.m, alternatively corresponding functions from the GSW Toolbox)
- compute absolute velocity relative to bottom and surface, use either $v_{ref} = 0$ or $\frac{\omega}{f} k(\nabla \times \eta)$, and compare

Exercise 3: Comparison to Numerical GCM Results

- load $\eta$ and velocity fields of numerical GCM ($ecco2_data.mat$)
- compare to corresponding fields of Exercise 1 and 2
Jürgen Kusche, Annette Eicker and Ehsan Forootan

Analysis Tools for GRACE- and Related Data Sets

Theoretical Basis

Lecture Notes


September 6, 2011

DFG Priority Program SPP1257
Mass Transport and Mass Distribution in the Earth System

Bonn University
These lecture notes were compiled on the occasion of the summerschool 'Global Hydrological Cycle', organized by DFG’s priority program SPP1257 'Mass Transport and Mass Distribution in the System Earth' at September 12-16, 2011 in Mayschoss/Ahr. Our aim was to familiarize students with different background (geodesy, hydrology, oceanography, geophysics) with some mathematical concepts that are fundamental for analysing level-2 products (sets of spherical harmonic coefficients) from the GRACE mission and related geophysical data sets (model outputs in gridded form). The focus was on concepts, and technical proofs were avoided. Specific topics were filtering and basin averaging, and the application of the principal component analysis technique. Thanks for spotting typos go to Volker Klemann and Torsten Mayer-Gürr.

Jürgen Kusche, Annette Eicker and Ehsan Forootan

Bonn, September 6, 2011
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   1.2 Smoothing of Spherical Harmonic Models  
      1.2.1 Isotropic Filters  
      1.2.2 Anisotropic Filters  
   1.3 Smoothed Area Averaging  
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   3.2 Spherical Coordinates  

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Smoothing and Averaging of Functions on the Sphere

At the product level 2, GRACE data are condensed to monthly sets of fully normalized spherical harmonic coefficients. These coefficients are the outcome of a rather complex data processing scheme. For scientific analysis, users of GRACE data will have to perform a number of basic operations on the GRACE coefficients: transform dimensionless geopotential coefficients into gridded geoid heights or maps of surface density expressed through equivalent water heights, and average such maps over the surface of some hydrological catchment area or ocean basin. Moreover, one of the problems that users of the GRACE level-2 products face is the presence of increasing correlated noise ('stripes') at higher frequencies. Smoothing operators can be applied in either spatial or spectral domain in order to suppress the effect of noise in maps and area averages. The purpose of this chapter is to describe the mathematical concepts underlying these procedures.

Notation

According to [2], we write the gravitational potential at a fixed location as a function of time as

\[
V(r, \theta, \lambda, t) = \frac{GM}{r} + \frac{GM}{r} \sum_{n=2}^{\infty} \left( \frac{R}{r} \right)^n \sum_{m=0}^{n} P_{nm}(\cos \theta) \left( \tilde{C}_{nm}(t) \cos m\lambda + \tilde{S}_{nm}(t) \sin m\lambda \right)
\]

with \( \theta = \frac{\pi}{2} - \phi \), \( GM = \mu \) (in [2]), and fully normalized spherical harmonic coefficients \( \tilde{C}_{nm} \) and \( \tilde{S}_{nm} \). The other quantities will be explained later.

In the following, we will refer to either temporal variations in the geoid or in total water storage (TWS), the spherical harmonic coefficients of which we will denote as \( f_{nm} \). These quantities are related to the gravitational potential via simple spectral relations, which are valid under certain assumptions.
('spherical approximation', radial Earth model) that will be discussed elsewhere during the summer school. Under these assumptions, the geoid or TWS changes projected to the space domain read

\[ F = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{f}_{nm} \bar{Y}_{nm}(\lambda, \theta). \]

Total water storage change from geopotential harmonics. In this case, the common approximation is

\[ \bar{f}_{nm}(t) = R \rho_e \frac{2n + 1}{3} \frac{1}{1 + k'_n} (\bar{v}_{nm}(t) - \bar{v}_{nm}) \]

with

\[ \bar{v}_{nm} = \bar{C}_{nm} \quad \text{for} \quad m \geq 0, \quad \bar{v}_{nm} = \bar{S}_{|m|} \quad \text{for} \quad m < 0, \]

and \( \bar{v}_{nm} \) either a suitable long-term mean of these (i.e. \( \bar{v}_{nm} = \langle \bar{v}_{nm}(t) \rangle_{t_0} \)) or they refer to a reference epoch \( t_0 \).

In the above, TWS is expressed as a surface density (unit \( \text{kg m}^{-2} \)), the height of a water column is usually derived by scaling the above by a reference density of water, i.e. multiplication by \( 1/\rho_w \). Thus, \( \rho_w \) (if applied) is a reference quantity which has to be chosen as a convention (usually, \( \rho_w = 1000 \text{kg m}^{-3} \) or \( \rho_w = 1025 \text{kg m}^{-3} \)). The average density of the Earth, \( \rho_e \), is related to the Earth’s mass \( M \) by \( M = \frac{4\pi}{3} \rho_e R^3 \) and follows therefore to \( \rho_e = 5517 \text{kg m}^{-3} \). Finally, the \( k'_n \) are the elastic gravity load Love numbers and follow from 1D-models of the Earth’s rheologic properties.

![Fig. 1 Shown are the coefficients \( \frac{\rho_e 2n + 1}{3} \frac{1}{1 + k'_n} \). It is obvious that, when computing TWS harmonics from geopotential harmonics, errors in higher degrees will be amplified compared to those in low-degree coefficients.](image-url)
1.1 Area Averaging (‘Windowing’)

In many applications one is interested in averaging a field over a certain geographical area or region or basin. This is what we call area averaging or windowing here.

1.1.1 Exact Windowing

Let us start with a given field $F$ defined on the sphere $\Omega$,

$$F = F(\lambda, \theta)$$  \hspace{1cm} (1.1)

expressed either in spatial domain or in spherical harmonic representation.

The area $O \subset \Omega$ can be mathematically expressed through its characteristic function

$$O = O(\lambda, \theta) = \begin{cases} 1 & (\lambda, \theta) \in O \\ 0 & (\lambda, \theta) \notin O \end{cases}$$  \hspace{1cm} (1.2)

(1 inside the area, and 0 outside of it). Its area (size) $\bar{O}$ is

$$\bar{O} = \int_{O} d\omega = \int_{\Omega} O d\omega .$$  \hspace{1cm} (1.3)

Windowing $F$ over $O$ means to derive an average

$$\bar{F}_O = \frac{1}{\bar{O}} \int_{O} F d\omega = \frac{1}{\bar{O}} \int_{\Omega} OF d\omega$$  \hspace{1cm} (1.4)

of $F$ over $O$.

Remark. If $F = F(\lambda, \theta, t)$ is a spatio-temporal field, the average $\bar{F}_O(t)$ will be a time-series.

Remark. Usually, a polygon $O(\lambda_i, \theta_i), i = 1 \ldots q$ will be used to characterize $O(\lambda, \theta)$.

Remark. In practical computations in the space domain, the integrals will be replaced by discrete sums, introducing a discretization error $\epsilon$ whose magnitude depends on the spatial grid resolution and the smoothness of both the function $F$ and the region boundary of $O$. The discrete version of Eq. (1.4) can be cast as $\bar{F}_O = o^T f$ or $\bar{F}_O(t) = o^T f(t)$ if the field $F$ is time-dependent.

Remark. If $F$ is approximated by a spherical harmonic expansion before projecting onto a grid and evaluation of the discrete sum, an extra truncation error is introduced.
1.1.2 Exact Windowing in Spherical Harmonic Representation

Next, we consider both $F$ and $O$ expanded in $4\pi$-normalized spherical harmonics $\bar{Y}_{nm}$.

\[
F = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{f}_{nm} \bar{Y}_{nm}(\lambda, \theta) \tag{1.5}
\]

\[
O = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{O}_{nm} \bar{Y}_{nm}(\lambda, \theta) \tag{1.6}
\]

(the overbar in $\bar{f}_{nm}$ etc. tells that coefficients are $4\pi$-normalized). It is immediately clear that

\[
\bar{O} = \int_{\Omega} O \bar{Y}_{00} d\omega = 4\pi \bar{O}_{00} \tag{1.7}
\]

The exact average of $F$ over $O$ is then

\[
\bar{F}_O = \frac{1}{\bar{O}_{00}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{O}_{nm} \bar{f}_{nm} \tag{1.8}
\]

or, if $\bar{o}_{nm} = \frac{\bar{O}_{nm}}{\bar{O}_{00}}$ are the region’s SH coefficients further normalized to $\bar{o}_{00} = 1$,

\[
\bar{F}_O = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{o}_{nm} \bar{f}_{nm} \tag{1.9}
\]

Example. The first $4\pi$-normalized coefficients of the ocean function are given in the table.

<table>
<thead>
<tr>
<th>$n$ $m$</th>
<th>$\bar{O}_{nm}$</th>
<th>$\bar{O}_{n-m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0</td>
<td>0.701227 · 10^6</td>
<td>0.000000 · 10^6</td>
</tr>
<tr>
<td>1 0</td>
<td>-0.176689 · 10^6</td>
<td>0.000000 · 10^6</td>
</tr>
<tr>
<td>1 1</td>
<td>-0.115778 · 10^6</td>
<td>0.635533 · 10^{-1}</td>
</tr>
<tr>
<td>2 0</td>
<td>0.618996 · 10^{-2}</td>
<td>0.000000 · 10^0</td>
</tr>
<tr>
<td>2 1</td>
<td>-0.450010 · 10^{-1}</td>
<td>-0.717864 · 10^{-1}</td>
</tr>
<tr>
<td>2 2</td>
<td>0.471078 · 10^{-1}</td>
<td>0.464998 · 10^{-2}</td>
</tr>
<tr>
<td>3 0</td>
<td>-0.355365 · 10^{-2}</td>
<td>0.000000 · 10^0</td>
</tr>
<tr>
<td>3 1</td>
<td>0.518058 · 10^{-1}</td>
<td>-0.251440 · 10^{-1}</td>
</tr>
<tr>
<td>3 2</td>
<td>0.691439 · 10^{-1}</td>
<td>-0.992945 · 10^{-1}</td>
</tr>
<tr>
<td>3 3</td>
<td>-0.135222 · 10^{-1}</td>
<td>-0.947600 · 10^{-1}</td>
</tr>
</tbody>
</table>

Remark. In practical computations, the spherical harmonic summation will
be evaluated up to finite degree \( \bar{n} \), and a truncation error results. The exact average of \( F \) over \( O \) can be split into

\[
\bar{F}_O = \frac{1}{O_{00}} \sum_{n=0}^{\bar{n}} \sum_{m=-n}^{n} \bar{O}_{nm} \bar{f}_{nm} + \frac{1}{O_{00}} \sum_{n=\bar{n}+1}^{\infty} \sum_{m=-n}^{n} \bar{O}_{nm} \bar{f}_{nm} \, .
\]  

(1.10)

This may be written as \( \bar{F}_O = \bar{o}^T \bar{f} + \epsilon \). The second term is the truncation or omission error. It will vanish if either \( F \) or \( O \) is band-limited with degree \( \bar{n} \), or if the high-degree components of \( F \) and \( O \) are orthogonal on the sphere in \( L_2 \)-sense.

**Remark.** If \( F \) is truncated at degree \( \bar{n} \), then projected into the space domain and the integral is evaluated over an exactly delineated area, the above-mentioned truncation error will occur as well.

### 1.2 Smoothing of Spherical Harmonic Models

Smoothing or filtering a field is usually applied to suppress ‘rough’ or ‘oscillatory’ or ‘noisy’ components.

Consider \( F \) according to Eq. (1.1) and Eq. (1.5). A smoothed version is obtained by convolving \( F \) against a two-point kernel \( W \) with suitable properties.

In the spatial domain,

\[
F_W(\lambda, \theta) = \int_\Omega W(\lambda, \theta, \lambda', \theta') F(\lambda', \theta') d\omega
\]

(1.11)

and in the spectral domain, in the rather general case of an arbitrarily shaped window function,

\[
\bar{F}_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{f}_{nm} \bar{Y}(\lambda, \theta) , \quad \bar{f}_{nm} = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \bar{w}_{nm'} \bar{f}_{n'm'} \bar{Y}_{n'm'}(\lambda, \theta)
\]

(1.12)

Thus, \( W(\lambda, \theta, \lambda', \theta') \) describes the weighted contribution of \( F \) at point \( \lambda, \theta \) to the windowed function \( F_W \) at point \( \lambda', \theta' \). In its discrete version in either spatial or spectral domain, smoothing will read \( \bar{f}_W = Wf \) (up to a truncation error, which we will omit in the sequel). The \( \bar{f}_{nm} \) are the SH coefficients of the smoothed version of \( F \). And the \( \bar{w}_{nm'} \) are the \( 4\pi \)-normalized spherical harmonic coefficients of the two-point smoothing kernel

\[
W(\lambda, \theta, \lambda', \theta') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \bar{w}_{nm'} \bar{Y}_{nm}(\lambda, \theta) \bar{Y}_{n'm'}(\lambda', \theta')
\]

(1.13)

Or, \( W(\lambda, \theta, \lambda', \theta') = \bar{y}^T(\lambda, \theta) W \bar{y}(\lambda', \theta') \) with matrix \( W \) containing the elements \( \bar{w}_{nm'} \). It is obvious that for fixed \( \lambda', \theta' \) the \( \bar{w}_{nm'} \bar{Y}_{n'm'}(\lambda', \theta') \) are the
Smoothing and Averaging of Functions on the Sphere

4π-normalized spherical harmonic coefficients of \( W(\lambda, \theta) \), and vice versa; for fixed \( \lambda, \theta \) the \( w_{nm} \) are the SH coefficients of \( W(\lambda, \theta) \). Consequently, for a given two-point kernel \( W \),

\[
\tilde{w}_{nm} = \frac{1}{(4\pi)^2} \int_{\Omega} \int_{\Omega'} W(\lambda, \theta, \lambda', \theta') \tilde{Y}_{nm}(\lambda, \theta) \tilde{Y}_{nm}(\lambda', \theta') d\omega d\omega'.
\]

This is the most general case of smoothing.

1.2.1 Isotropic Filters

Most filters that we encounter in the literature are isotropic, i.e. the smoothing kernel depends only on the spherical distance \( \psi \) between the two points \( \lambda, \theta \) and \( \lambda', \theta' \) and not on their relative orientation. A comprehensive review is [4]. For isotropic kernels, the SH coefficients of the kernel can be reduced to the Legendre coefficients of a zonal (z-symmetric) function \( w_n \). Thus,

\[
W(\psi) = \sum_{n=0}^{\infty} (2n + 1) w_n P_n(\cos \psi)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n Y_{nm}(\lambda, \theta) \bar{Y}_{nm}(\lambda', \theta') = W(\lambda, \theta, \lambda', \theta')
\]

(with \( P_n(\cos \psi) \) being the unnormalized Legendre polynomials) or

\[
\tilde{w}_{nm} = \delta_{nm} w_n .
\]

For isotropic filters, smoothing of \( F \) can be written as

\[
F_W(\lambda, \theta) = \int_{\Omega} W(\psi) F(\lambda', \theta') d\omega
\]

and in the spectral domain simply \( f_{nm}^W = w_n f_{nm} \) and

\[
F_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n \bar{f}_{nm} \bar{Y}(\lambda, \theta) .
\]

Example. A first example is the boxcar filter, which simply truncates the function \( F \) at SH degree \( \bar{n} \)

\[
W(n)(\psi) = \sum_{n=0}^{\bar{n}} (2n + 1) P_n(\cos \psi), \quad w(n) = \begin{cases} 1 & \text{for } n \leq \bar{n} \\ 0 & \text{for } n > \bar{n} \end{cases}
\]

Example. A second example is the Gaussian filter, popularized by [19], for which we know an analytic expression in the spatial domain ('bell-shaped') with
1.2 Smoothing of Spherical Harmonic Models

\[ W_d(\psi) = 2b \frac{e^{-b(1-\cos \psi)}}{1 - e^{-2b}} = \sum_{n=0}^{\infty} (2n+1)w_n^{(d)}P_n(\cos \psi) \]

with

\[ b = \frac{\ln(2)}{1 - \cos \frac{dR}{R}}. \]

Here, \( d = R\psi_d \) is the 'half-width' radius parameter where the kernel drops from 1 at \( \psi = 0 \) to \( \frac{1}{2} \), which is commonly used to indicate the degree of smoothing. The Legendre coefficients of the Gaussian filter are found from recursion relations,

\[ w_0^{(d)} = 1, \quad w_1^{(d)} = \left( 1 + \frac{e^{-2b}}{1 - e^{-2b}} - \frac{1}{b} \right), \quad w_{n+1}^{(d)} = -\frac{2n+1}{b}w_n^{(d)} + w_{n-1}^{(d)}. \]

![Fig. 2](image.png)

**Fig. 2** Shown are the Legendre coefficients \( w_n^{(d)} \) for \( d \) equal to 100 km, 500 km and 1000 km.

### 1.2.2 Anisotropic Filters

Anisotropic filters can be characterized into symmetric (or diagonal) filters and non-symmetric filters ([14]). A further differentiation among non-symmetric filters is possible when thinking of the coefficients \( \bar{w}_{nm}' \) as being ordered within matrix \( \bar{W} \) (when we use a particular ordering scheme for the \( \bar{f}_{nm} \), the same has to apply to the filter coefficients).

For *symmetric* filters, \( \bar{W} \) is diagonal and

\[ \bar{w}_{nm}' = \delta_{nm} \bar{w}_{nm}. \]  

(1.19)

Thus the smoothing kernel has the shape

\[ W(\lambda, \theta, \lambda', \theta') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{w}_{nm} \bar{Y}_{nm}(\lambda, \theta)\bar{Y}_{nm}(\lambda', \theta'). \]  

(1.20)
It is symmetric with respect to the points \( \lambda, \theta \) and \( \lambda', \theta' \).

**Example.** Han’s filter ([11]) is of this type. They chose a Gaussian filter with ‘order-dependent’ smoothing radius \( d(m) \),

\[
w_{nm} = w_n^{(d(m))} , \quad d(m) = \frac{d_1 - d_0}{m_1} m + d_0
\]

**Example.** The ‘fan’ ([21]) filter is simply a product of different Gaussian filters applied to order and degree,

\[
w_{nm} = w_{nm}^{(d_1, d_2)} = w_n^{(d_1)} w_m^{(d_2)}
\]

In the general case of Eq. (1.12), the filter is *non–symmetric* with respect to points \( \lambda, \theta \) and \( \lambda', \theta' \) and its matrix \( W \) is full.

**Remark.** Even if \( W \) is symmetric, the resulting filter would be *non–symmetric.*

**Example.** The DDK filter ([15], [16]). Here, the filter matrix is derived by regularization of a ‘characteristic’ normal equation system that involves a-priori information on the signal variance and the observation system from which we obtain the unfiltered coefficients,

\[
W_{(\alpha)} = L_{0}N = (N + \alpha M)^{-1}N
\]

or

\[
\overline{w}_{nm}^{(\alpha)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} L_{nm}^{(\alpha)} N_{mm}
\]

with \( M \) being an approximation to \( C_f = E\{\hat{f}\hat{f}^T\} \), \( N \) being an approximation to \( C_{\hat{f}} = E\{\hat{ff}^T\} \), and \( L_{nm}^{(\alpha)} \), \( N_{mm}^{(\alpha)} \) the corresponding elements. In addition, \( \alpha \) is a damping parameter. The DDK filters are non-symmetric. In [16] it was shown that the original \( W_{(\alpha)} \) of [15] can be safely replaced by a block-diagonal version of the matrix.

**Example.** The 'Swenson and Wahr' filter ([20]) is non-symmetric and it can also be represented by a block-diagonal \( W \). The idea of this filter is that an empirical model for the correlations between SH coefficients \( \bar{f}_{nm} \) of the same order and parity is formulated and then used for decorrelation.

**Example.** EOF filtering means one applies PCA to a time series \( f_i \) of either gridded values of \( F \) or SH coefficients. A reconstruction with \( q \) modes provides

\[
f^{(q)} = EI^{(q)}E^T \bar{f} = W^{(q)} f
\]

where \( E \) contains the EOFs of the time series and \( I^{(q)} \) is a diagonal matrix with unity in the first \( q \) entries and zero otherwise. EOF filtering corresponds to application of a non-symmetric filter as well.
1.3 Smoothed Area Averaging

Now let us come back to the windowing of a spherical harmonic model $F$, i.e. we wish to average $F$ over the region $O$. Of course we can window a smoothed version $F_W$ of $F$ as well, if necessary.

Another view on the same operation is as follows: In place of Eq. (1.2), we may introduce a smoothed area function $O_W$,

$$O_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tilde{O}_W^{nm} Y_{nm}(\lambda, \theta) = \frac{1}{4\pi} \int_{\Omega} W(\lambda, \theta, \lambda', \theta') O(\lambda', \theta') d\omega' \ ,$$

and we will apply $O_W$ to the original function $F$

$$\bar{F}_{O_W} = \frac{1}{\bar{O}_W} \int_{\Omega} O_W F d\omega \quad (1.22)$$

(note that $\bar{O}_W = \bar{O}$ if the filter is normalized, see below). In general, the smoothing kernel $W$ is a two-point function on the sphere, cf. Eq. (1.3).

1.3.1 Spherical Harmonic Representation

In case of Eq. (1.15), i.e. $W$ is isotropic, the smoothed area function can be written as

$$O_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tilde{O}_W^{nm} Y_{nm}(\lambda, \theta) \quad (1.23)$$

with

$$\tilde{O}_W^{nm} = \bar{w}_n \bar{O}_{nm} \ . \quad (1.24)$$

The smoothed area average is found in the spectral domain as

$$\bar{F}_{O_W} = \frac{1}{\bar{O}_W^{00}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n \bar{O}_{nm} \bar{f}_{nm} \ . \quad (1.25)$$

The choice $w_0 = \bar{w}_0^{00} = 1$ ('filter normalization') guarantees that

$$\frac{1}{4\pi} \int_{\Omega} O_W d\omega = \bar{O}_W^{00} = \bar{O}_{00} = \frac{1}{4\pi} \int_{\Omega} O d\omega \quad . \quad (1.26)$$

I.e. the 'area' of the smoothly varying window $O_W$ equals to the area of $O$.

But, the smoothing kernel will inevitably 'leak' energy beyond the original region. I.e.

$$\frac{1}{4\pi} \int_{\Omega} O_W d\omega = \frac{1}{4\pi} \int_{O} O_W d\omega + \frac{1}{4\pi} \int_{\Omega/O} O_W d\omega \ . \quad (1.27)$$

The above can be transferred to the more general case of non-isotropic smoothing without any problem.
1.3.2 Amplitude Damping ('Bias')

For a given area $O$, windowing or smoothing will decrease the amplitude of the average $\bar{F}_W$ with respect to the original average $\bar{F}$. What causes this reduction is best understood by explicitly writing down the 'reduction factor' $\beta_{O,W,F}$, which we define as

$$\beta_{O,W,F} = \frac{\bar{F}_W}{\bar{F}_O}$$

and which is specific for a certain area $O$, a certain window kernel $W$, and an input function $F$. For an isotropic smoothing kernel,

$$\beta_{O,W,F} = \frac{\bar{O}}{\bar{O}_0} \int_{\Omega} \frac{O_W F d\omega}{O F d\omega} = \frac{1}{w_0} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n \bar{O}_{nm} \bar{f}_{nm}$$

and for $w_0 = 1$

$$\beta_{O,W,F} = 1 - \frac{1}{\bar{O}_{00}} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (1 - w_n) \bar{O}_{nm} \bar{f}_{nm}$$

The reduction factor clearly depends on the basin shape, the filter coefficients, and the signal itself.

Example. For $w_0 = 1$ and $F = c$, where $c$ is a constant (i.e. the signal is constant over the whole sphere), $\beta$ is exactly one, i.e. no damping occurs at all.

Example. For $w_0 = 1$ and $F = c \cdot O(\lambda, \theta)$ (the signal is constant over the area $O$, and exactly zero outside), the damping factor becomes (considering $\int_{\Omega} O^2 d\omega = \int_{\Omega} O d\omega = \bar{O}_{00}$)

$$\beta_{O,W,c \cdot O} = 1 - \frac{1}{c \cdot \bar{O}_{00}} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (1 - w_n) \bar{O}_{nm}^2$$

Example. In ([15]), a 'standard damping factor' ('scaling bias') is defined for smoothing a constant signal over a spherical cap area, and numbers are provided for Gaussian and DDK filters of different degree of smoothing and at different geographical latitudes.

1.4 Filter Shape

1.4.1 Impulse Response

For comparing smoothing kernels in the spatial domain, it is helpful to map a kernel’s impulse response. This can be best understood when we imagine an
area \( O \) shrinks to a point on the sphere. By letting the basin function degrade to a Dirac function (we want to see the smoothing effect for a particular location \( \lambda', \theta' \)), we obtain

\[
O^\delta(\lambda, \theta) = \delta^{\lambda', \theta'}(\lambda, \theta) = \begin{cases} 
\infty & \text{for } \lambda' = \lambda, \theta' = \theta \\
0 & \text{otherwise}
\end{cases} 
\quad (1.31)
\]

and

\[
O^\delta_{nm} = \frac{1}{4\pi} \int_{\Omega} \delta^{\lambda', \theta'}(\lambda'', \theta'') Y_{nm}(\lambda'', \theta'') d\omega = \bar{Y}_{nm}(\lambda', \theta') . \quad (1.32)
\]

**Remark.** Eq. (1.32) is very helpful in practical applications, since one only has to compute the \( \bar{Y}_{nm}(\lambda', \theta') \). Or, with the spherical harmonic representation of the Dirac,

\[
O^\delta(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{Y}_{nm}(\lambda', \theta') \bar{Y}_{nm}(\lambda, \theta) . \quad (1.33)
\]

Consequently, the impulse response of the most general non-isotropic two-point kernel \( W \) will be

\[
O^\delta_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} w^{n', m'}_{nm} \bar{Y}_{n'm'}(\lambda', \theta') \bar{Y}_{nm}(\lambda, \theta) . \quad (1.34)
\]

And for an isotropic kernel

\[
O^\delta_W(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w_n \bar{Y}_{nm}(\lambda', \theta') \bar{Y}_{nm}(\lambda, \theta) . \quad (1.35)
\]

**1.4.2 Localization**

The localization of an isotropic smoothing kernel can be best measured by its 'half-with' radius, i.e. the distance \( d = R\psi_d \) where the kernel drops from 1 at \( \psi = 0 \) to \( \frac{1}{2} \)

\[
W(\psi_d) = \frac{1}{2} . \quad (1.36)
\]

For non-isotropic kernels, measuring the localization is more difficult. Unlike with isotropic kernels, it will depend on the particular location \( \lambda', \theta' \). There, one might compute the half-with radius in two directions - North and East.

Following [17] and[1], [15] introduced the variance \( \sigma_W \) of the squared normalized window function \( W(\lambda, \theta) \) at location \( \lambda', \theta' \) as a single measure for its localization properties. The variance is the second centralized moment of a probability density function defined on the sphere; it is an integral measure for the spreading about the expectation and it is independent of introducing a particular coordinate system on the sphere.
We suppose with [1] that $W^2$ has been normalized,

$$
\int_{\Omega} W^2(\lambda', \theta') d\omega = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \bar{w}_{nm}^{'m}{Y}_{nm}(\lambda', \theta') \right)^2 = 1 .
$$  \hfill (1.37)

The integration in the first term applies to $\lambda, \theta$. Normalization is required in order to interpret $W$ as a probability density function. The expectation in the space domain is introduced ([1]) via

$$
\mu_W = \int_{\Omega} e W^2 d\omega
$$  \hfill (1.38)

where $e = (\sin \theta \cos \lambda, \sin \theta \sin \lambda, \cos \theta)^T$ is the unit vector pointing from the origin to a location on $\Omega$. If $W^2(\lambda, \theta)$ (for given $\lambda', \theta'$) is thought to represent a surface density distribution, $\mu_W$ points to its center of mass (which is inside of $\Omega$).

As the unit vector can be represented through the unnormalized degree-1 spherical harmonics

$$
e = (Y_{11}, Y_{1-1}, Y_{10})^T
$$  \hfill (1.39)

we can write the components of $\mu_W$ as

$$
\mu_{W;x} = \int_{\Omega} W^2 Y_{11} d\omega = (W^2)_{11} \quad \mu_{W;y} = (W^2)_{1-1} \quad \mu_{W;z} = (W^2)_{10} .
$$  \hfill (1.40)

The variance of $W(\lambda, \theta)$ is introduced in the usual fashion, i.e. as the expectation of $(e - \mu_W)^2$

$$
\sigma_W^2 = \int_{\Omega} (e - \mu_W)^2 W^2 d\omega
$$  \hfill (1.41)

Because of $(e - \mu_W)^2 = 1 + (\mu_W)^2 - 2e^T \mu_W$ and $\int -2e^T \mu_W W^2 d\omega = -2(\mu_W)^2$, the variance is simply

$$
\sigma_W^2 = 1 - (\mu_W)^2 = 1 - \sum_{m=-1}^{1} (W^2)_{1m}^2 .
$$  \hfill (1.42)

and its computation requires only the computation of the degree–1 harmonics of $W^2$.

The degree–1 harmonics of $W^2$ may be computed directly, involving the Clebsch-Gordon coefficients, or simply by projecting the normalized $W^2$ onto a grid and subsequent spherical harmonic analysis.
Principal Component Analysis and Related Ideas

Products of geodetic observing systems (GRACE, altimetry) and geophysical modelling are most often represented in form of time series of spatial maps (total water storage, sea level anomalies,...). The user of these products will often find a few spatial pattern dominating the variability within these maps. Identifying these patterns can aid in physical interpretation, comparison of different data sets, and removing unnecessary small-scale signals or noise. Eigenspace techniques as the principal component method are among the most popular analysis techniques supporting these objectives. The purpose of this chapter is to describe the mathematical concepts behind the principal component analysis (PCA), to introduce some alternative formulations, and to make the reader aware of some of the many choices to be made by the analyst.

2.1 Principle Component Analysis

2.1.1 PCA as a Data Compression Method: Mode Extraction and Data Reconstruction

Sampling spatio-temporal fields can lead to huge amounts of data. For example, a field observed or modelled on a $1^\circ \times 1^\circ$ grid, with a time step of one day, provides already more than $23 \cdot 10^6$ data elements for one year of data. It is now a challenging task to reduce the dimensionality of the data vector and to identify the most important patterns explaining the variability of the system. The Empirical Orthogonal Functions (EOF) technique, also called Principal Component Analysis (PCA), has become one of the most widely used methods. General references are [6] and [7]. In pattern analysis, PCA is also known as Karhunen-Loeve transform or Hotelling transform.

PCA has been used extensively to extract individual dominant modes of the data variability, while simultaneously suppressing those modes connected with
Principal Component Analysis and Related Ideas

low variability and therefore reducing the number of data efficiently. The physical interpretability of the obtained pattern (i.e. in terms of independent physical processes) is, however, a point of discussion as the obtained modes are by definition orthogonal in space and time and this is not necessarily so in reality.

Consider the \( n \times 1 \) data vector \( y_i \), given for \( p \) time epochs \( t_i \),

\[
y_i = \begin{pmatrix}
y_{1;i} \\
y_{2;i} \\
\vdots \\
y_{n;i}
\end{pmatrix} \quad i = 1 \ldots p .
\]

Typically, \( y_i \) contains the values of an observed or modelled field in \( n \) locations (the nodes of a two-dimensional grid or a set of discrete scattered observation sites; but the \( y_i \) could also contain \( n \) spherical harmonic coefficients), at time \( t_i \). We will assume that the data are centered, i.e. the time average per node

\[
\frac{1}{p} \sum_{i=1}^{p} y_{j;i} = 0
\]

Another way to look at eq. (2.1) is to decompose the data vector \( y_i = I y_i \), according to the individual locations,

\[
y_i = y_{1;i} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + y_{2;i} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + y_{n;i} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = y_{1;i} u_1 + y_{2;i} u_2 + \cdots + y_{n;i} u_n .
\]

The basis vectors \( u_j \) are independent of time, orthogonal, normalized with respect to the standard scalar product \( (a, b) = a^T b \), and they are each associated with an individual location. One may interpret the original observations \( y_{j;i} \) as coordinates in an “observation space” with regard to the trivial unit basis \( u_j \), in an \( n \)-dimensional vector space. Clearly, this interpretation suggests that other bases and other coordinates might be useful as well. The following will lead to a different choice of basis.

We collect all \( y_i \) in the \( n \times p \) data matrix \( Y \) (assuming in what follows that the data is complete in the sense that for every location \( j \) there exists a data value \( y_{j;i} \) for any epoch \( t_i \)). With other words, we assume for every location in the set there exists an uninterrupted time series of observations. The data matrix is then
2.1 Principle Component Analysis

\[ Y = (y_1, y_2, \ldots, y_p) = \begin{pmatrix} y_{1;1} & y_{1;2} & \cdots & y_{1;p} \\ y_{2;1} & y_{2;2} & \cdots & y_{2;p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n;1} & y_{n;2} & \cdots & y_{n;p} \end{pmatrix}, \quad (2.4) \]

Its rows contain the time series per location, whereas its columns contain the entire data from all locations per time epoch.

We might be weighting the data matrix, e.g. taking the individual accuracy of the data at different locations into account, or according to the latitude of the nodes. In this case, the homogeneized data matrix becomes \( \bar{Y} = YG \), where \( GG^T = P \) is the weight matrix.

The \( n \times n \) signal covariance matrix \( C \) contains the variances and covariances (i.e. second central moments) of the data viewed as time series per location. From the data samples \( y_i \), it can be estimated (empirically) as

\[ C = \frac{1}{p} YY^T = \frac{1}{p} \begin{pmatrix} \sum_{i=1}^{p} y_{1;i}^2 & \sum_{i=1}^{p} y_{1;i} y_{2;i} & \cdots & \sum_{i=1}^{p} y_{1;i} y_{n;i} \\ \sum_{i=1}^{p} y_{2;i} y_{1;i} & \sum_{i=1}^{p} y_{2;i}^2 & \cdots & \sum_{i=1}^{p} y_{2;i} y_{n;i} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{p} y_{n;i} y_{1;i} & \sum_{i=1}^{p} y_{n;i} y_{2;i} & \cdots & \sum_{i=1}^{p} y_{n;i}^2 \end{pmatrix}, \quad (2.5) \]

or using the weighted matrix \( \bar{Y} \) instead. Note that the signal covariance matrix \( C' = \frac{1}{n} \bar{Y}^T \bar{Y} \), in contrast, contains the spatial variance and covariances of the data viewed as a function of position, for any \( t_i \); there the sum extends over the \( n \) locations. Adding all the \( n \) individual variances from the time series provides what is often called the total variance,

\[ \Delta^2 = \frac{1}{p} \sum_{j=1}^{n} \left( \sum_{i=1}^{p} y_{j;i}^2 \right) = \text{trace}(C). \quad (2.6) \]

An alternative way to decompose the data vector is given by the eigenvalue decomposition of the signal covariance matrix \( C \)

\[ C = E \Lambda E^T \quad (2.7) \]

where \( \Lambda \) is a diagonal matrix containing the \( n \) eigenvalues \( \lambda_i \), and the columns of the orthogonal \( n \times n \) matrix \( E \) contain the corresponding eigenvectors \( e_i \). The sum of all eigenvalues equals to the matrix trace, and therefore to the total variance

\[ \sum_{j=1}^{n} \lambda_j = \Delta^2. \quad (2.8) \]

We assume the eigenvalues and eigenvectors are ordered according to the magnitude of the eigenvalues; i.e. \( \lambda_1 \) is the largest one. Then, one can state that each eigenvalue “explains” a fraction
 Principal Component Analysis and Related Ideas

\[ \eta_j = \frac{\lambda_j}{\Delta^2} \]  

(2.9)

of the total variance, with the first eigenvalue explaining the largest part and so on. The eigenvalues of \( \mathbf{C} = \frac{1}{p} \mathbf{Y} \mathbf{Y}^T \) equal to \( \frac{1}{\sqrt{p}} \) times the singular values of the data matrix \( \mathbf{Y} \). The SVD of the data matrix can be written

\[ \mathbf{Y} = \mathbf{E} \mathbf{\Delta} \mathbf{D}, \]  

(2.10)

where of course now

\[ \lambda_j = \frac{1}{p} \Delta^2_j \]  

(2.11)

We will come back later to the \( n \times n \) diagonal matrix \( \Delta \) and the \( n \times p \) orthogonal matrix \( \mathbf{D} \).

Principal component analysis replaces the basis \( \mathbf{u}_j \) by the eigenvectors \( \mathbf{e}_j \) of \( \mathbf{C} \) as the vector basis for representing the original observations \( \mathbf{y}_i \). One has to adopt a convention about the scaling of the eigenvectors, and in what follows we will assume they are normalized,

\[ \mathbf{e}_j^T \mathbf{e}_j = 1, \]  

(2.12)

and their first entry is positive

\[ e_{1,j} > 0 \]  

(2.13)

just as it was the case for the original basis \( \mathbf{u}_j \). In the same way as the \( \mathbf{u}_j \) can be associated with a discrete version of a delta function (they point exactly at the \( j \)-th data location with a value of one there, and zero values otherwise), the \( \mathbf{e}_j \) can be viewed as discrete version of a function which describes common pattern in the entire data. They are called empirical orthogonal functions (EOFs) or simply 'modes'. The first EOF \( \mathbf{e}_1 \) contains thus the dominant pattern (that is, if \( \lambda_1 \) is distinctly larger than the other eigenvalues). If the original data is provided on two-dimensional gridded locations, it is common to visualize the corresponding EOFs on this grid. Then, the principal component representation of the \( n \times 1 \) data vector at \( t_i, i = 1, \ldots, p \) is

\[ \mathbf{y}_i = d_{1,i} \mathbf{e}_1 + d_{2,i} \mathbf{e}_2 + \cdots + d_{n,i} \mathbf{e}_n = \sum_{j=1}^{n} d_{j,i} \mathbf{e}_j = \mathbf{E} \mathbf{d}_i \]  

(2.14)

where the “principal components” (PCs) or PC scores \( d_{j,i} \) are determined from projecting the original data onto the new basis

\[ d_{j,i} = \mathbf{e}_j^T \mathbf{y}_i. \]  

(2.15)

The \( d_{j,i} \) can be viewed upon as time series, \( i = 1 \ldots p \), whereas the index \( j \) points at the pattern \( \mathbf{e}_j \) where the time series is associated with. Or,
\[ \mathbf{d}_i = \mathbf{E}^T \mathbf{y}_i . \] (2.16)

Since \( \mathbf{E}^T \mathbf{E} = \mathbf{I} \), this can be written as \( \mathbf{d}_i = (\mathbf{E}^T \mathbf{E})^{-1} \mathbf{E}^T \mathbf{y}_i \), as well. As the ordering is according to the magnitude of the eigenvalues, it is often sufficient to compute only a few, say \( \bar{n} \), of the \( d_{ji} \). The reconstructed data will then still exhibit the largest part of the total variability:

\[ \bar{\mathbf{y}}_i = \sum_{j=1}^{\bar{n}} d_{ji} \mathbf{e}_j . \] (2.17)

By this construction, EOFs constitute (normalized) spatial patterns whose amplitude evolution is given by the corresponding PC. The EOF itself does not change in time.

**Remark.** In other words, PCA decomposes the original data into time-invariant (‘standing’) spatial pattern, which are scaled by the corresponding time-variable PC. Therefore, PCA is not suitable for discovering propagating pattern in the data, since those will be distributed over several standing modes in the analysis.

**Remark.** Since the data are assumed as centered, one may say that PCA makes use of the second central moments of the data (only) to decorrelate them.

**Remark.** From the point of view of estimation theory, Eq. (2.5) assumes that the data are perfectly centered. In practice, one will probably compute and remove the sample mean of the time series. Then, in Eq. (2.5), one might use \( \frac{1}{p-1} \) in place of \( \frac{1}{p} \) in order to unbiasedly estimate the second central moments. It does not matter for the computation of the EOFs and the PCs, since the EOFs will be normalized (Eq. 2.12) anyway and the PCs follow from the normalized EOFs and the data.

**Remark.** The reconstructed data, Eq. (2.17) can be expressed by

\[ \bar{\mathbf{y}}_i = \mathbf{E} \mathbf{I}^{(n)} \mathbf{E}^T \mathbf{y}_i , \]

where \( \mathbf{I}^{(n)} \) is a diagonal matrix with unity in the first \( \bar{n} \) entries and zero otherwise, i.e., decomposition and partial reconstruction can be viewed as a linear operation (in first order at least).

From Eq. (2.14), it is clear that the data matrix \( \mathbf{Y} \) is referred to the EOFs by

\[ \mathbf{Y} = \mathbf{E} \mathbf{D} , \] (2.18)

where the rows of \( \mathbf{D} \) now contain the PCs for all EOFs (e.g., the first row contains the temporal evolution of the first EOF), and the columns of \( \mathbf{D} \) contain the PC vectors \( \mathbf{d}_i \) (each vector contains the temporal amplitude of all EOFs for one particular epoch). With other words, we write
Principal Component Analysis and Related Ideas

\[ D = \begin{pmatrix} d_{1;1} & d_{1;2} & \cdots & d_{1;p} \\ d_{2;1} & d_{2;2} & \cdots & d_{2;p} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n;1} & d_{n;2} & \cdots & d_{n;p} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} y_{j;1}^2 & \sum_{j=1}^{n} y_{j;1} y_{j;2} & \cdots & \sum_{j=1}^{n} y_{j;1} y_{j;p} \\ \sum_{j=1}^{n} y_{j;2} y_{j;1} & \sum_{j=1}^{n} y_{j;2}^2 & \cdots & \sum_{j=1}^{n} y_{j;2} y_{j;p} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} y_{j;p} y_{j;1} & \sum_{j=1}^{n} y_{j;p} y_{j;2} & \cdots & \sum_{j=1}^{n} y_{j;p}^2 \end{pmatrix}. \]

Then, for the total variance

\[ \Delta^2 = \text{trace} (E D D^T E^T) = \text{trace} (D^T D^T) = \sum_{j=1}^{n} \sum_{i=1}^{p} d_{j;i}^2. \] (2.19)

The aim of PCA is to find a linear combination of the original data nodes that explains the maximum variability (variance) of the data. This means, we are searching for the mode \( e \) such that \( Y e \) has maximum variance. The variance of the centered time series \( Y e \) is

\[ \frac{1}{p} (Y e)^T (Y e) = \frac{1}{p} e^T C e. \] (2.20)

Usually we require \( e \) to be normalized. The task is then to maximize Eq. (2.20) subject to \( e^T e = 1 \). The solution to this problem is the eigenvalue problem \( C e = \lambda e \), with eigenvectors \( e_i \) and eigenvalues \( \lambda_i \) as introduced earlier.

However, the data vectors \( y_i \) will contain a random error, and such will the eigenvalues and eigenvectors derived from the data matrix. This has to be considered in particular if eigenvalues are close to each other.

2.1.2 Temporal PCA versus Spatial PCA

PCA as described above is sometimes called temporal PCA, since it departs from the correlations between time series of data (which are contained in the \( n \times n \) covariance matrix \( C \)). On the other hand, it is perfectly valid to consider, for the same data set, the spatial correlations and built the \( p \times p \) spatial covariance matrix \( C' = \frac{1}{n} Y^T Y \), or

\[ C' = \frac{1}{n} Y^T Y = \frac{1}{n} \begin{pmatrix} \sum_{j=1}^{n} y_{j;1}^2 & \sum_{j=1}^{n} y_{j;1} y_{j;2} & \cdots & \sum_{j=1}^{n} y_{j;1} y_{j;p} \\ \sum_{j=1}^{n} y_{j;2} y_{j;1} & \sum_{j=1}^{n} y_{j;2}^2 & \cdots & \sum_{j=1}^{n} y_{j;2} y_{j;p} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} y_{j;p} y_{j;1} & \sum_{j=1}^{n} y_{j;p} y_{j;2} & \cdots & \sum_{j=1}^{n} y_{j;p}^2 \end{pmatrix}. \] (2.21)

In fact, if \( p \ll n \), storing \( C' \) requires much less memory space compared to storing \( C \).

PCA based upon \( C' \) is called spatial PCA. Of course, temporal and spatial PCA are closely related: \( C \) and \( C' \) are of different dimension but they share the same eigenvalues (apart from a factor that depends only on \( n \) and \( p \)).
An eigenvalue decomposition (and comparison with the decomposition of $C$) reveals

$$C' = \frac{p}{n} \bar{D}^T \bar{D}$$

(2.22)

where we have $\bar{D} = \Delta^{-1} E^T Y = \Delta^{-1} D$.

It is thus obvious that the $k$-th EOF of the spatial PCA ($k$-th column of $\bar{D}^T$) corresponds to the $k$-th PCs of the temporal PCA. Alternatively, this can be seen as follows: From

$$Ce_j = \lambda_j e_j$$

follows

$$\frac{n}{np} Y^T Y Y^T e_j = \lambda_j Y^T e_j$$

and the eigenvectors of $C'$ can be read off as $Y^T e_j$. Thus

$$E' = Y^T = D = \bar{D}$$

### 2.1.3 PCA of Linearly Transformed Data

It is interesting to consider the PCA of a set of linearly transformed $m \times 1$ data vectors

$$z_i = Ay_i, \quad i = 1 \ldots p$$

(2.23)

with $m \times n$ matrix $A$. Again $p$ is the number of time epochs. The number of data nodes $m$ might be larger, equal or less than $n$.

**Example.** The original data might contain spherical harmonic coefficients of a field, and the transformed data contain gridded values. In this case $m > n$ is not uncommon. Matrix $A$ contains the spherical harmonics for each given coefficient evaluated for each grid node.

**Example.** The original data contain values on a global grid of certain spacing. We ask in how far the EOFs and PCs on a local subgrid, i.e. for some region of the globe, will differ from those evaluated from the global data set. In this case, $m < n$ and the matrix $A$ equals to the identity matrix, with its rows removed for all nodes that are not present in the local subgrid.

Obviously the transformed data matrix is $Z = AY$. Furthermore we have

$$C_z = \frac{1}{p} ZZ^T = \frac{1}{p} AYY^T A^T = AE\Lambda E^T A^T$$

(2.24)

where $E$ and $A$ contain the eigenvectors and eigenvalues of the original data covariance matrix. Obviously, the eigenvectors and eigenvalues of $C_z$ will differ from those of $C$, meaning that both the EOFs and the PCs of the transformed data will differ from those of the original data (unless in some special cases).

Let $\mu_i$ be the eigenvalues of $A^T A$. For the eigenvalues of $C_z = ACA^T$, which equal to the eigenvalues of $CA^T A$, the following inclusion holds ([5])
2.1.4 PCA as a Data Whitening Method: Homogeneization

Obviously, one can interpret the PCs as a ‘whitened’ version of the original data. To make this clear, we will consider instead of

\[ d_i = E^T y_i \]  

(2.26)

the homogeneized PCs \( \bar{d}_{j,i} = \frac{1}{\sqrt{\lambda_i}} d_{j,i} \), or

\[ \bar{d}_i = \Lambda^{-\frac{1}{2}} d_i = \Lambda^{-\frac{1}{2}} E^T y_i = \bar{E}^T y_i . \]  

(2.27)

Here, we have introduced the column-by-column scaled matrix \( \bar{E} = E \Lambda^{-\frac{1}{2}} \).

Remark. It is clear by now that the homogeneized PCs \( \bar{d}_i \) are the column vectors of the SVD matrix \( \bar{D} \).

The scaled EOFs are not of unit length anymore, but still orthogonal,

\[ \bar{E}^T \bar{E} = \Lambda^{-1} . \]  

(2.28)

The signal covariance matrix of the original data \( y_i \) is \( C = E \Lambda E^T \), thus the covariance of the PCs will be

\[ C_d = E^T E \Lambda E E^T = \Lambda . \]  

(2.29)

or, for clarity,

\[ \sum_{i=1}^{p} d_{j,i}^2 = \lambda_j . \]

And the signal covariance of the homogeneized PCs will be

\[ C_{\bar{d}} = \Lambda^{-\frac{1}{2}} E^T E \Lambda E E^T \Lambda^{-\frac{1}{2}} = I . \]  

(2.30)

From the last two expressions, it is obvious that the PCs and the homogeneized PCs are uncorrelated, with the latter ones also being of unit variance. Therefore, the (homogeneized) PCA is often viewed as a data whitening transformation. The homogeneized EOFs are directly obtained from applying the rescaling to the original eigenvectors

\[ \bar{e}_j = \frac{1}{\sqrt{\lambda_j}} e_j , \]  

(2.31)

of the data.
2.1.5 Number of Modes

In many applications of PCA, we will avoid to retain all \( n \) modes, but rather use a subset of \( \bar{n} \) dominant ones. The reasoning can be different: We may want to compress the data, or we may want to get rid of those modes that supposedly contain noise. Or, PCA is just considered as a preprocessing and we will subsequently apply e.g. rotation on the dominant modes. Let \( \mathcal{J} = \{j_1, j_2, \ldots, j_{\bar{n}}\} \) denote the index set of all modes to be retained, i.e.

\[
\bar{y}_i = \sum_{j \in \mathcal{J}} d_{j_i} e_j .
\]

A rule that determines \( \mathcal{J} \) is called a selection rule.

It has been suggested by Eq. (2.9) that each eigenvalue of the data covariance explains a certain fraction of the total variance \( \Delta^2 \), Eq. (2.6). This indicates that the strategy to choose a reasonable subset of modes could simply be

\[
\mathcal{J} = \{ j \mid \sum_{j \in \mathcal{J}} \eta_j > \epsilon \} .
\]

This strategy is by far the most often followed one, with a typical threshold value of 0.9.

A selection rule (North’s rule) that is often considered goes back to [18]. It is based on the perception that the data \( y_i \) represent independent realizations or samples of a random field with unknown stochastic moments. From these realizations, one will be able to reconstruct the true covariance \( C' \) only up to an error that depends on \( C' \) and the number \( n \) of data realizations. With other words, \( C \) as computed through Eq. (2.5) will be considered as a stochastic quantity being contaminated by an error whose covariance can be estimated from \( C \) and \( n \). Therefore, the eigenvalues and eigenvectors of \( C \) have to be considered as stochastic as well. [18] proceed to show that ‘typical’ errors of neighbouring eigenvalues and eigenvectors will then be

\[
\delta \lambda_j = \sqrt{\frac{2}{n} \lambda_j} + \cdots \quad \delta e_j = \frac{\delta \lambda_j}{\lambda_k - \lambda_j} e_k + \cdots
\]

‘Neighbouring’ means that \( \lambda_k \) is the eigenvalue numerically closest to \( \lambda_j \). This selection rule says that if the ‘typical’ error of an eigenvalue is comparable to the difference of this eigenvalue to its neighbour, then the ‘typical’ error of the corresponding EOF will be of the size of the neighbouring EOF itself. One will then tend to disregard this mode in the reconstruction. Or,

\[
\mathcal{J} = \{ j \mid |\delta \lambda_j| < |\lambda_k - \lambda_j| = \min_{i \neq j} |\lambda_i - \lambda_j| \} .
\]

Several other selection rules have been proposed since then, based on different principles. More recently, Monte Carlo methods have been applied frequently to test the statistical significance of modes.
2.1.6 PCA as a Tool for Comparing Multiple Data Sets

We are often interested in comparing multiple data sets, e.g. satellite-derived vs. modelled, or different model output data sets. Several statistical algorithms allow to derive correlation measures, similarities and joint pattern and so on. Here, we will only focus on the application of the PCA as described before in such a situation.

Consider the $n \times 1$ vector $y_i$, given for $p$ time epochs $t_i$, and extracted from $M$ different data sets, or

$$y_i^{(m)} = \left( y_{1,i}^{(m)}, y_{2,i}^{(m)}, \ldots, y_{n,i}^{(m)} \right) \quad i = 1 \ldots p, \quad m = 1 \ldots M ,$$

(2.32)

which we may recast in a 'super data matrix'

$$X = \left( Y^{(1)}, Y^{(2)}, \ldots, Y^{(m)} \right) .$$

(2.33)

If all data vectors are considered as equally good, bias-free (i.e. centered free of errors), and describing the same phenomena apart from unavoidable data/model errors, i.e. as independent realizations of the same data vector, one may simply compute the covariance matrix

$$C = \frac{1}{pM} XX^T$$

(2.34)

and go on as described before.

If we suspect that different sensors or models see different phenomena, which is to say the data are not coming from the same p.d.f., one may of course apply PCA on each data set independently. This provides $M$ covariance matrices $C^{(m)}$. A comparison is then hampered by the fact that each data set will be represented in its own basis $e_j^{(m)}$. To facilitate comparison, one may project all data sets onto the basis derived from $C$ or from one of the data sets (maybe the one we trust most), say, from $C^{(m^*)}$. This is, we compare the data sets on the level of principle components with a joint basis,

$$d_i^{(m)} = E^T y_i^{(m)}$$

(2.35)

or

$$d_i^{(m)} = E^{(m^*)T} y_i^{(m)} .$$

(2.36)
2.1 Principle Component Analysis

2.1.7 Rotation

Rotated EOF is a technique which attempts to overcome some common shortcomings of PCA. For example, the mathematical constraints (orthogonality of EOFs and uncorrelatedness of PCs) of PCA, in connection with the dependence of the computation domain (see 'PCA of Linearly Transformed Data') may render the modes found in data difficult to interpret. Physical modes may not necessarily be orthogonal and thus leak into several different mathematical modes in PCA. REOF is a technique which sacrifices either orthogonality of the EOFs or uncorrelatedness of the PCs, while adding new optimization criteria that seek to find physically plausible modes.

Rotated homogeneized EOFs

An understanding of the idea of REOF starts with the observation that, viewed as a whitening transformation, PCA with the basis vectors $\bar{e}_j$ is not unique. To see this, the data vectors $y_i$, with covariance $C$ are expressed by

$$y_i = E \Lambda^{\frac{1}{2}} d_i .$$  \hfill (2.37)

It is possible to replace the $\bar{d}_i$ by any set of $i = 1, \ldots, p$ rotated $n \times 1$ homogeneized PCs,

$$\bar{r}_i = V \bar{d}_i ,$$  \hfill (2.38)

with $n \times n$ orthogonal matrix $V$, i.e. $V^T V = I$. Then,

$$C_r = V V^T = I .$$  \hfill (2.39)

We have

$$\bar{d}_i = V^T \bar{r}_i$$  \hfill (2.40)

and

$$y_i = E \Lambda^{\frac{1}{2}} V^T \bar{r}_i = E \Lambda V^T \bar{r}_i .$$  \hfill (2.41)

It is obvious that the data covariance $C = E \Lambda^{\frac{1}{2}} V^T (E \Lambda^{\frac{1}{2}} V^T)^T = E \Lambda E^T$ does not depend on $V$. Hence, the transform $y_i = E \Lambda^{\frac{1}{2}} V^T \bar{r}_i = E \Lambda V^T \bar{r}_i$ with rescaled and rotated PCs whitens the data as good as the original homogeneized PCs. The rotated basis vectors (or rotated EOFs) are now the column vectors of $\bar{F} = E \Lambda^{\frac{1}{2}} V^T = E \Lambda V^T$.

We have seen in Eq. (2.40) that the rotated homogeneized PCs have diagonal and equal covariance, just as the original homogeneized PCs,

$$C_r = C_d = I .$$

The PCs, viewed as time series per EOF, are uncorrelated and they do not lose this property when an arbitrary orthogonal rotation is applied to the EOFs. The rotated homogeneized EOFs, however, will not be orthogonal anymore,

$$\bar{F}^T \bar{F} = V \Lambda^{\frac{1}{2}} E^T E \Lambda^{\frac{1}{2}} V^T = V \Lambda V^T .$$  \hfill (2.42)
Rotated EOFs

On the other hand, one can define rotated EOFs by straight application of an orthogonal matrix $V$ to the EOFs $E$,

$$ F = EV^T $$

(2.43)

i.e. without homogeneizing the PCs first. The data is then represented through rotated PCs,

$$ y_i = F^T r_i. $$

(2.44)

In this case, the rotated EOFs remain orthogonal, since

$$ F^T F = V E^T E V^T = I. $$

(2.45)

But now, the $i = 1, \ldots, p$ rotated PCs

$$ r_i = V d_i = F^T y_i $$

(2.46)

loose the property of being uncorrelated since

$$ C_r = V A V^T. $$

(2.47)

In summary, by rotation either the orthogonality of the EOFs or the uncorrelatedness of the PCs will be destroyed.

Rotation principles

So far, nothing has been said regarding the particular choice of an orthogonal matrix $V$ in EOF and PC rotation. All orthogonal $V$ are able to reproduce the data, whereas only for $V = I$ both orthogonality in space and time can be preserved. Which one (in space or time) we sacrifice by rotation, depends upon application to homogeneized or original EOFs and PCs.

In REOF, one usually specifies an optimization criterion $F(V)$ in terms of rotated EOFs or rotated PCs, to be met subject to the condition $VV^T = I$. In other words, an orthogonal $n \times n$ matrix has $n(n-1)/2$ degrees of freedom and these have to be chosen such as to optimize $F(V)$.

When we have

$$ F = EV^T $$

with elements $f_{j,i}$ of the $j$th rotated EOF, the following family of VARIMAX criteria is in use

$$ F(V) = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} f_{j,i}^4 - \frac{\gamma}{n} \left( \sum_{j=1}^{n} f_{j,i}^2 \right)^2 \right). $$

(2.48)
The quantity inside the summation is proportional to the variance of the square of the rotated EOFs $f_j$ (for $\gamma = 1$). This variance will be big if some values $f_{ji}$ are close to 1 and many are near 0. Consequently, it is often claimed that the varimax rotation attempts to 'simplify' the patterns by localizing the 'regions of action'.

In practice, one will rotate only the first $\bar{n}$ EOFs corresponding to the largest singular values, then the above reads

$$
\mathcal{F}(\mathbf{V}) = \sum_{i=1}^{\bar{n}} \left( \sum_{j=1}^{n} f_{j;i}^4 - \frac{2}{n} \left( \sum_{j=1}^{n} f_{j;i}^2 \right)^2 \right) .
$$

(2.49)

2.2 Independent Component Analysis

We follow [12]. Consider orthogonal EOF rotation with homogeneized PCs, i.e.

$$
\bar{r}_i = \mathbf{V} \bar{d}_i
$$

(2.50)

for $i = 1, \ldots p$ time steps. Collecting the $n \times 1$ vectors of homogeneized PCs in $n \times$ matrices $\bar{D}$ and $\bar{R}$, this is

$$
\bar{R} = \mathbf{V} \bar{D} ,
$$

(2.51)

and the rotated EOFs will be

$$
\bar{F} = \mathbf{E} \Lambda^{1/2} \mathbf{V} .
$$

(2.52)

For any orthogonal $\mathbf{V}$ the rotated homogeneized PCs $\bar{r}_i$ are uncorrelated and of unit variance, i.e. as a time series in $i$

$$
\sum_{i=1}^{p} \bar{r}_{j;i}^2 = 1 \quad j = 1, \ldots, n
$$

and

$$
\sum_{i=1}^{p} \bar{r}_{j;i} \bar{r}_{k;i} = 0 \quad j \neq k
$$

In [12] it is suggested to choose $\mathbf{V}$ such that the $\bar{r}_i$ are close to being independent.

Independence is stronger than uncorrelatedness, and defining (and testing) it requires to involve higher moments of the pdf of the $\bar{r}_{j;i}$. Different criteria are in use in the literature on Independent Component Analysis (ICA).

The line of reasoning in [12] is as follows. If $\bar{r}_{j;i}$ and $\bar{r}_{k;i}$ are independent, then the time series of the squares $\bar{r}_{j;i}^2$, $\bar{r}_{k;i}^2$ should be uncorrelated (after centering), or
\[
\sum_{i=1}^{p} \left( r_{j,i}^2 - \frac{1}{p} \sum_{l=1}^{p} r_{j,l}^2 \right) \left( r_{k,i}^2 - \frac{1}{p} \sum_{l=1}^{p} r_{k,l}^2 \right) = 0 \quad j \neq k.
\]

This can be written in matrix notation. Let \( \odot \) denote the Hadamard matrix product, i.e.
\[
\bar{R} \odot \bar{R} = \begin{pmatrix}
r_{1;1}^2 & r_{1;2}^2 & \ldots & r_{1;p}^2 \\
r_{2;1}^2 & r_{2;2}^2 & \ldots & r_{2;p}^2 \\
\vdots & \vdots & \ddots & \vdots \\
r_{n;1}^2 & r_{n;2}^2 & \ldots & r_{n;p}^2
\end{pmatrix}
\]

and let \( H = H^2 \) be the \( p \times p \) centering matrix (with \( i = (1, 1, \ldots, 1)^T \))
\[
H = I - \frac{1}{p} ii^T.
\]

Then, for independent time series \( \bar{r}_{j,i} \) the (empirical) covariance matrix of the centered squares
\[
C_{r^2} = \frac{1}{p} (\bar{R} \odot \bar{R})H (\bar{R} \odot \bar{R})H^T = \frac{1}{p} (\bar{R} \odot \bar{R})H (\bar{R} \odot \bar{R})^T
\]

must be diagonal.

In other words, an ICA approach can be constructed by defining an objective function \( F(V) \) that penalizes off-diagonal elements of \( C_{r^2} \). ICA will then seek a rotation matrix \( V \) through optimization of \( F(V) \).

Remark. The above idea ([12]) makes use of fourth statistical moments, but other moments may be used for defining an objective function as well.
Appendix

3.1 Spherical Harmonics

Spherical harmonic series

It is common to represent real-valued phenomena on the sphere as spherical harmonic series

\[ F(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\cos \theta) \]  

(3.1)

with longitude \( \lambda \), colatitude \( \theta \), the spherical harmonic degree \( n \) and order \( m \), where \( n \geq m \geq 0 \), the spherical harmonic coefficients \( C_{nm} \) and \( S_{nm} \), and the associated Legendre functions of the first kind \( P_{nm} \).

Legendre polynomials and associated Legendre functions

The associated Legendre functions of degree \( n \) and order \( m \), \( n \geq m \geq 0 \), can be expressed through the \( m \)-th derivatives of the Legendre polynomials of degree \( n \), \( P_n = P_{n0} \), with respect to \( t = \cos \theta \),

\[ P_{nm}(t) = \left(1 - t^2\right)^{m/2} \frac{d^m P_n(t)}{dt^m}, \]  

(3.2)

which may be written as

\[ P_{nm}(\cos \theta) = \sin^m \theta \frac{d^m P_n(\cos \theta)}{d(\cos \theta)^m}. \]  

(3.3)

They fulfill the differential equation

\[ (1 - t^2) \frac{d^2 P_{nm}}{dt^2} - 2t \frac{dP_{nm}}{dt} + \left(n(n+1) - \frac{m^2}{1 - t^2}\right) P_{nm} = 0 \]  

(3.4)

or
\[
\frac{d}{d\theta} \left( \sin \theta \frac{dP_{nm}}{d\theta} \right) + \left( n(n+1) \sin \theta - \frac{m^2}{\sin \theta} \right) P_{nm} = 0.
\] (3.5)

Note that sometimes (e.g. [3]) the associated Legendre functions are defined as
\[P_{nm}^m = (-1)^m P_{nm}.\] The Rodrigues formula expresses the Legendre polynomials \(P_n\) of degree \(n\) through the \(n\)-th derivatives of \((1 - t^2)^n = \sin^{2n} \theta\),
\[P_n(t) = \frac{1}{2^n n!} \frac{d^n (t^2 - 1)^n}{dt^n} \] (3.6)

they satisfy the differential equation
\[n(n+1)P_n - 2t \frac{dP_n}{dt} + (1 - t^2) \frac{d^2P_n}{dt^2} = 0\] (3.7)
or
\[n(n+1)P_n + \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP_n}{d\theta} \right) = 0.\] (3.8)

An expansion of the Legendre polynomials and associated Legendre functions

| Table 3.1. Legendre polynomials and associated Legendre functions |
|-------------------|-------------------|
| \(n\) | \(m\) | \(P_{nm}\) |
| 0 | 0 | 1 |
| 1 | 0 | \(\cos \theta\) |
| 1 | 1 | \(\sin \theta\) |
| 2 | 0 | \(\frac{1}{4}(3 \cos^2 \theta - 1) = \frac{1}{4}(3 \cos 2\theta + 1)\) |
| 2 | 1 | \(3 \sin \theta \cos \theta = \frac{3}{2} \sin 2\theta\) |
| 2 | 2 | \(3 \sin^2 \theta\) |
| 3 | 0 | \(\frac{1}{4}(5 \cos^3 \theta - 3 \cos \theta) = \frac{1}{4}(5 \cos 3\theta + 3 \cos \theta)\) |
| 3 | 1 | \(\sin \theta(\frac{5}{4} \cos^2 \theta - \frac{3}{4}) = \frac{3}{4} \sin \theta(5 \cos 2\theta + 3)\) |
| 3 | 2 | \(15 \sin^2 \theta \cos \theta = \frac{15}{4} \sin \theta \sin 2\theta\) |
| 3 | 3 | \(15 \sin^3 \theta\) |

into trigonometric series reads

\[P_{nm}(\cos \theta) = \sin^m \theta \sum_{q=0}^{\text{int}(\frac{n-m}{2})} T_{nmq} \cos^{n-m-2q} \theta ,\] (3.9)

where \(\text{int}(x)\) means the integer part of \(x\), and the coefficients \(T_{nmq}\) are given by ([10],[9])
\[T_{nmq} = \frac{(-1)^q(2n-2q)!}{2^q q!(n-q)!(n-m-2q)!}.\] (3.10)

Relations (3.2) and (3.2) can be combined to
3.1 Spherical Harmonics

\[ P_{nm}(t) = \frac{(1-t^2)^{m/2}}{2^n n!} \frac{d^{n+m}(t^2-1)^n}{dt^{n+m}}. \]  

This is being used to define associate Legendre functions \( P_{nm} \) of negative order \( m; 0 > m \geq -n \). The relation between \( P_{nm} \) and \( P_{n,-m} \) is ([8])

\[ P_{n,-m}(t) = (-1)^m \frac{(n-m)!}{(n+m)!} P_{nm} \]  

\[ P_{nm}(t) = (-1)^m \frac{(n+m)!}{(n-m)!} P_{n,-m}. \]  

**Alternative notations for the real-valued spherical harmonic series**

There are \( 2n+1 \) spherical harmonics of degree \( n \). Another way to write eq. (3.1) is

\[ F(\lambda, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm} Y_{nm}(\lambda, \theta) \]  

with \( f_{nm} = C_{nm} \) for \( m \geq 0 \), \( f_{nm} = S_{n|m} \) for \( m < 0 \), and

\[ Y_{nm}(\lambda, \theta) = Y_{nm1}(\lambda, \theta) = \cos m \lambda \ P_{nm}(\cos \theta) \quad m \geq 0 \]  

\[ Y_{nm}(\lambda, \theta) = Y_{n|m|2}(\lambda, \theta) = \sin |m| \lambda \ P_{n|m|}(\cos \theta) \quad m < 0. \]  

**Integration over the unit sphere**

The spherical harmonics \( Y_{nm} \) are orthogonal on the unit sphere \( \Omega \). Integrating products of spherical harmonics \( Y_{nm} \) yields

\[ \int_{\Omega} Y_{nm} Y_{n'm'} d\omega = 4\pi \frac{1}{H_{nm}} \delta_{nm} \delta_{m'm'}. \]  

with

\[ H_{nm} = \sqrt{(2 - \delta_{0m})(2n + 1) \frac{(n-m)!}{(n+m)!}}, \]  

in particular

\[ H_{n0} = \sqrt{(2n + 1)}. \]  

Consequently,

\[ \int_{\Omega} Y_{nm} d\omega = 4\pi \delta_{n0} \delta_{m0}. \]  

Integrals over various products of derivatives of spherical harmonics can be found in [13].
It is common in geodesy to introduce $4\pi$- or fully normalized associated Legendre functions

\[ \bar{P}_{nm} = \Pi_{nm} P_{nm}. \] (3.21)

The relation between the $P_{nm}$ of positive order, $n \geq m \geq 0$ and those of negative order, $P_{n,-m}$, is

\[ \bar{P}_{n,-m}(t) = (-1)^m \bar{P}_{nm}(t), \] (3.22)

\[ \bar{P}_{nm}(t) = (-1)^m \bar{P}_{n,-m}. \] (3.23)

Using the $\bar{P}_{nm}$ of positive order, we introduce $4\pi$- or fully normalized spherical harmonics

\[ \bar{Y}_{nm} = \Pi_{nm} Y_{nm}. \] (3.24)

or

\[ \bar{Y}_{nm}(\lambda, \theta) = \cos m\lambda \bar{P}_{nm}(\cos \theta) \quad m \geq 0 \] (3.25)

\[ \bar{Y}_{nm}(\lambda, \theta) = \sin |m| \bar{P}_{|n|m}(|\cos \theta|) \quad m < 0. \]

with spherical harmonic coefficients $\bar{C}_{nm} = \frac{1}{\pi_{nm}} C_{nm}$, $\bar{S}_{nm} = \frac{1}{\pi_{nm}} S_{nm}$, or $\bar{f}_{nm}$, $\bar{f}_{nm1}$, $\bar{f}_{nm2}$ accordingly. By definition, these fully normalized spherical harmonics fulfill

\[ \int_{\Omega} \bar{Y}_{nm} \bar{Y}_{n'm'} d\omega = 4\pi \delta_{nn'} \delta_{mm'}. \] (3.26)

The addition theorem relates fully ($4\pi$-) normalized spherical harmonics and the (un–normalized) Legendre polynomials

\[ \frac{1}{2n+1} \sum_{m=-n}^{n} \bar{Y}_{nm}(\lambda, \theta) \bar{Y}_{nm}(\lambda', \theta') = P_n(\cos \psi). \] (3.27)

In particular,

\[ \frac{1}{2n+1} \sum_{m=-n}^{n} \bar{Y}_{nm}^2(\lambda, \theta) = 1. \] (3.28)

**Practical computation of the fully normalized spherical harmonics**

In practice, the fully normalized associated Legendre functions $P_{nm}(\cos \theta)$ are computed via recursion relations.

**Example.** One of the most often applied recursive algorithms for the normalized Legendre functions as a function of co-latitude $\theta$ is the following
c = cos θ
s = sin θ

\[ P_{00} = 1 \]
\[ P_{11} = \sqrt{3} \cdot s \]

\[ \text{do } n = 2, \bar{n} \]
\[ a_n = \sqrt{\frac{2n + 1}{2n}} \]
\[ \bar{P}_{nn} = a_n \cdot s \cdot P_{n-1,n-1} \]
\[ \text{end do} \]

\[ \text{do } n = 1, \bar{n} \]
\[ b_n = \sqrt{2n + 1} \]
\[ \bar{P}_{nn-1} = b_n \cdot c \cdot P_{n-1,n-1} \]
\[ \text{end do} \]

\[ \text{do } m = n, 0, -1 \]
\[ c_n = \sqrt{\frac{(2n + 1)}{(n - m)(n + m)}} \]
\[ d_n = \sqrt{2n - 1} \]
\[ e_n = \sqrt{\frac{(n - m - 1)(n + m - 1)}{(2n - 3)}} \]
\[ \bar{P}_{nm} = c_n \cdot (d_n \cdot c \cdot P_{n-1,m} - e_n \cdot P_{n-2,m}) \]
\[ \text{end do} \]

\[ \text{end do} \]

**Normalized complex spherical harmonics**

Normalized complex spherical harmonics are introduced in different ways. Following e.g. [8] and using associated Legendre functions of positive and negative order, \( n \geq m \geq -n \)

\[ \bar{Y}_{nm} = \frac{(-1)^m}{\sqrt{4\pi}} \Xi_{nm} (\cos m\lambda + i \sin m\lambda) P_{nm}(\cos \theta) \]

\[ = \frac{(-1)^m}{\sqrt{4\pi}} \Xi_{nm} e^{im\lambda} P_{nm}(\cos \theta) \]

where

\[ \Xi_{nm} = \sqrt{\frac{(2n + 1)(n - m)!}{(n + m)!}} = \frac{\Pi_{nm}}{\sqrt{2 - \delta_{mn}}} \]

\[ \text{(3.30)} \]
Here
\[ \bar{Y}_{nm} = (-1)^m \bar{Y}_{n,-m}, \quad \bar{Y}_{nm}^* = (-1)^m \bar{Y}_{n,-m} \] (3.31)
follows from eq. (3.23) and
\[ \Xi_{nm} P_{nm} (\cos \theta) (\cos m \lambda + i \sin m \lambda) = (-1)^m \Xi_{n,-m} P_{n,-m} (\cos \theta) (\cos m \lambda - i \sin (-m \lambda)). \]

Consequently, in place of eq. (3.29) we could write
\[ \bar{Y}_{nm} = (-1)^m \sqrt{4\pi} \Xi_{nm} (\cos m \lambda + i \sin m \lambda) P_{nm} (\theta) \quad m \geq 0 \] (3.32)
\[ = (-1)^m \bar{Y}_{n|m|} \quad m < 0. \]

This is to relate complex spherical harmonics of negative order to associated Legendre functions of positive order. The \( \bar{Y}_{nm} \) are 1-normalized, thus
\[ \int_\Omega \bar{Y}_{nm} \bar{Y}_{nm'}^* d\omega = \delta_{nn'} \delta_{mm'} . \] (3.33)

And,
\[ \sum_{m=-n}^{n} \bar{Y}_{nm} (\lambda, \theta) \bar{Y}_{nm'} (\lambda', \theta') \]
\[ = \bar{Y}_{n0} (\lambda, \theta) \bar{Y}_{n0}^* (\lambda', \theta') + \sum_{m=1}^{n} (\bar{Y}_{nm} (\lambda, \theta) \bar{Y}_{nm}^* (\lambda', \theta') + \bar{Y}_{nm} (\lambda, \theta) \bar{Y}_{nm} (\lambda', \theta')) \]
\[ = \frac{1}{4\pi} \sum_{m=-n}^{n} \bar{Y}_{nm} (\lambda, \theta) \bar{Y}_{nm} (\lambda', \theta') = \frac{2n+1}{4\pi} P_n (\cos \psi). \] (3.35)

The relation between the complex \( \bar{Y}_{nm} \) and the real-valued valued \( \bar{Y}_{nm} \) is thus
\[ \bar{Y}_{nm} = \frac{(-1)^m}{\sqrt{4\pi} \sqrt{2 - \delta_{0m}}} (\bar{Y}_{nm} + i \bar{Y}_{n,-m}) \quad m \geq 0 \]
\[ \bar{Y}_{nm} = \frac{1}{\sqrt{4\pi} \sqrt{2}} (\bar{Y}_{n|m|} - i \bar{Y}_{n,-|m|}) \quad m < 0. \]

Some useful integrals are expressed below, using both unnormalized and fully normalized spherical harmonic representation.
\[ \frac{1}{4\pi} \int_\Omega F d\omega = f_{00} = \bar{f}_{00} \] (3.36)
\[ \frac{1}{4\pi} \int_\Omega F^2 d\omega = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{f_{nm}^2}{P_{nm}^2} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_{nm}^2 \] (3.37)
\[ \frac{1}{4\pi} \int_{\Omega} F G \, d\omega = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{f_{nm} g_{nm}}{P_n^2} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm} g_{nm} \]  \hfill (3.38)

\[ \frac{1}{4\pi} \int_{\Omega} F Y_{nm} \, d\omega = \frac{f_{nm}}{P_n^2} = \frac{\tilde{f}_{nm}}{P_n^2} \]  \hfill (3.39)

\[ \frac{1}{4\pi} \int_{\Omega} F \bar{Y}_{nm} \, d\omega = \tilde{f}_{nm} \]  \hfill (3.40)

\[ \frac{1}{4\pi} \int_{\Omega} F \tilde{Y}_{nm} \, d\omega = \frac{(-1)^m}{\sqrt{4\pi}} \frac{1}{\sqrt{2 - \delta_{0m}}} \left( f_{nm} + i \tilde{f}_{nm} \right) \quad m \geq 0 \]

\[ = \frac{(-1)^m}{\sqrt{4\pi}} \frac{1}{\sqrt{2}} \left( f_{n|m|} + i \tilde{f}_{n,-|m|} \right) \quad m < 0 \]  \hfill (3.41)

### 3.2 Spherical Coordinates

We use spherical longitude \( \lambda \), co-latitude \( \theta = \frac{\pi}{2} - \phi \) and radius \( r \). 'Geodetic' coordinates can all be easily transformed to spherical coordinates.

A vector field, when represented with respect to the local basis \( e_r, e_\theta, e_\lambda \), reads

\[ \mathbf{f} = f_r e_r + f_\theta e_\theta + f_\lambda e_\lambda . \]  \hfill (3.42)

The gradient and the Laplace operator applied to a 3D-function \( F(\lambda, \theta, r) \) in spherical coordinates are

\[ \nabla F = \frac{\partial F}{\partial r} e_r + \frac{1}{r} \frac{\partial F}{\partial \theta} e_\theta + \frac{1}{r \sin \theta} \frac{\partial F}{\partial \lambda} e_\lambda . \]  \hfill (3.43)

\[ \Delta F = \nabla \cdot \nabla F = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial F}{\partial \theta} \right) \]  \hfill (3.44)

\[ + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \lambda^2} \]

\[ = \frac{\partial^2 F}{\partial r^2} + \frac{2}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r^2 \tan \theta} \frac{\partial F}{\partial \theta} \]

\[ + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \lambda^2} \]

The gradient of the vector field can be written as a matrix, with entries

\[ \nabla \mathbf{f} = \begin{pmatrix}
\frac{\partial f_r}{\partial r} & \frac{\partial f_r}{\partial \theta} & \frac{f_r}{r} \\
\frac{\partial f_\theta}{\partial r} & \frac{\partial f_\theta}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial f_\theta}{\partial \lambda} - \frac{f_\theta}{r} \\
\frac{\partial f_\lambda}{\partial r} & \frac{\partial f_\lambda}{\partial \theta} + \frac{f_\lambda}{r} & \frac{1}{r \sin \theta} \frac{\partial f_\lambda}{\partial \lambda} - \cot \theta \frac{f_\lambda}{r}
\end{pmatrix} \]  \hfill (3.45)

The divergence of the vector field is
$$\mathbf{\nabla} \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 f_r\right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta f_\theta\right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda} f_\lambda$$ (3.46)

$$= \frac{2}{r} f_r + \frac{\partial}{\partial r} f_r + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta f_\theta\right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda} f_\lambda$$

For completeness, we note the strain tensor $\mathbf{\epsilon} = \frac{1}{2}(\mathbf{\nabla} \mathbf{u} + (\mathbf{\nabla} \mathbf{u})^T)$ and the (Cauchy) stress tensor in spherical coordinates:

$$\mathbf{\epsilon} = \epsilon_{rr} \mathbf{e}_r \mathbf{e}_r^T + \epsilon_{\theta\theta} \mathbf{e}_\theta \mathbf{e}_\theta^T + \epsilon_{\lambda\lambda} \mathbf{e}_\lambda \mathbf{e}_\lambda^T + \epsilon_{r\theta} \left(\mathbf{e}_r \mathbf{e}_\theta^T + \mathbf{e}_\theta \mathbf{e}_r^T\right) + \epsilon_{r\lambda} \left(\mathbf{e}_r \mathbf{e}_\lambda^T + \mathbf{e}_\lambda \mathbf{e}_r^T\right) + \epsilon_{\theta\lambda} \left(\mathbf{e}_\theta \mathbf{e}_\lambda^T + \mathbf{e}_\lambda \mathbf{e}_\theta^T\right)$$ (3.47)

in particular

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$ (3.48)

$$\epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} u_r$$ (3.49)

$$\epsilon_{\lambda\lambda} = \frac{1}{r \sin \theta} \frac{\partial u_\lambda}{\partial \lambda} + \frac{1}{r} u_\theta + \frac{1}{r \tan \theta} u_\theta$$ (3.50)

$$\epsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r} u_\lambda\right)$$ (3.51)

$$\epsilon_{r\lambda} = \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \lambda} + \frac{\partial u_\lambda}{\partial r} - \frac{1}{r} u_\lambda\right)$$ (3.52)

$$\epsilon_{\theta\lambda} = \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \lambda} + \frac{1}{r \tan \theta} u_\lambda\right)$$ (3.53)

and

$$\mathbf{\sigma} = \sigma_{rr} \mathbf{e}_r \mathbf{e}_r^T + \sigma_{\theta\theta} \mathbf{e}_\theta \mathbf{e}_\theta^T + \sigma_{\lambda\lambda} \mathbf{e}_\lambda \mathbf{e}_\lambda^T + \sigma_{r\theta} \left(\mathbf{e}_r \mathbf{e}_\theta^T + \mathbf{e}_\theta \mathbf{e}_r^T\right) + \sigma_{r\lambda} \left(\mathbf{e}_r \mathbf{e}_\lambda^T + \mathbf{e}_\lambda \mathbf{e}_r^T\right) + \sigma_{\theta\lambda} \left(\mathbf{e}_\theta \mathbf{e}_\lambda^T + \mathbf{e}_\lambda \mathbf{e}_\theta^T\right).$$ (3.54)
References

13. Higgins TP, Kopal Z (1968) Volume integrals of the products of spherical harmonics and their application to viscous dissipation phenomena in fluids, Astrophysics and space science 2:352-360
Surface Loading

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Version of 06/09/2011
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Part I

Surface loads

In this lecture I will present the basic concept of a load applied to the earth surface and its interaction with the earth’s interior. Consider these notes as a draft.

1 Mathematical prerequisites

1.1 Definition of surface load

A mass at the surface of the earth is usually condensed to an infinitesimal layer at the reference height of the surface. Mass condensation results in representation of mass distribution as a surface mass density where the integral of density over the vertical reduces to a surface mass density measured in kg/m$^2$,

$$\sigma(a, \Omega) = \int_{r_0}^r \rho_\sigma(r, \Omega) \, dr .$$

There, $\rho_\sigma$ is the density of the considered mass, $r_0 = [a \pm \delta r]$ is its height range, $\Omega = \theta, \varphi$ is the coordinate pair on the sphere. Furthermore, for a surface mass, the height range should not differ too much from the reference height to which it is condensed; usually it coincides with the earth radius, $a = 6.371 \times 10^6$ m. Furthermore the 3d character is of no importance as long as we are not to near to the load. With respect to solid earth problems, this is of course justified by the thin surface shell covering the earth’s interior. In consequence, all redistribution of water, ice and vapor are considered as a surface mass. The total mass of the load is not considered but its conservation

$$\int_{\Omega_0} \sigma d\Omega = 0 .$$

This is also stated in the title of the priority program, ‘mass distribution and mass transport in the earth system’.

1.2 Processes inducing surface loading

The main processes responsible for mass transport at the surface are

- hydrological water cycle
- ocean currents
- glacial melting
- atmospheric water cycle
These processes are coupled by the transport of water between the subsystems. This motivates the focus of the SPP. Other material like CO$_2$ or sediments are not considered. The loading of these processes will all deform the solid earth but, due to their different scales in dimension, amplitude and duration, they are considered differently.

1.3 Separation of signals

In this lecture the focus is on the response of the solid earth to surface load variations, which is detectable in geodetic observables, like GNSS and gravity. These observables represent integral responses to all relevant processes which can be of internal and external origin. The usual task for the geoscientist before interpreting the signal is to separate the signals of different processes. The classical approach for field measurements in order to detect a specific signal was to choose an appropriate location. Signals determined from GRACE represent integrals over large spatial areas. Therefore this strategy is only in parts successful. One alternative strategy is to separate the signals due to their temporal behaviour. The most important signals in this respect are seasonal, interannual or secular. There, the seasonal signals are quite simple to isolate, whereas the separation of interannual and secular variations is much more difficult. One of the most prominent secular signals is GIA proceeding on a time scale of kyr. It can be identified in gravity, surface displacement, rotational variations and in sea level. Other interannual signals which influence the mass redistribution are mostly of shorter time scales.

1.4 Solid earth response

One has of course to ask what is the response of the solid earth and how it is best modelled. Due to the mainly global scale of the considered loads, and the discussed response, we have to model the earth as a deformable sphere. With respect to the time scales involved for the ocean dynamics and atmosphere as for present day ice melting the earth is considered as an elastic body. But, the earth’s mantle reacts only on short time scales like an elastic solid, with the duration of the process, also anelasticity has to be considered and for very long time scales (thousands of years) the earth’s mantle behaves like a Newtonian fluid. Secular motions are therefore affected by both phenomena long-period mass redistributions at the surface and the visco-elastic response of the solid earth. The details will be discussed in Section 4, p. 14.

Basically for geodetic purpose, the response of the earth results in a displacement and a variation of the gravity potential considered at the earth’s surface. The latter is described as the change of the reference potential. These fields are usually...
represented by spherical-harmonics decomposition, i.e.
\[
\mathbf{u}(R, \Omega) = \sum_{lm} U_{lm}(R) \mathbf{S}^{-1}_{lm}(\Omega) + \sum_{lm} V_{lm}(R) \mathbf{S}^{+1}_{lm}(\Omega) + \sum_{lm} W_{lm}(R) \mathbf{S}^{0}_{lm}(\Omega)
\]
(3)
and
\[
\phi^{(1)}(R, \Omega) = \sum_{lm} \Phi_{lm}(R) Y_{lm}(\Omega)
\]
(4)
Here, \(U, V, W\) are radial functions describing the spheroidal displacement in radial and horizontal direction and the toroidal displacement, respectively, and \(\mathbf{S}^{(l,p)}\) are the corresponding vector spherical harmonics.\(^1\) The potential perturbation, \(\phi^{(1)}\), is represented in the same way by scalar spherical harmonics \(Y_{lm}\). In this formal representation the quantities are complex in order to keep a more compact representation. The relation to the real-valued \(4\pi\)-normalized spherical harmonics you will find in the appendix to the lecture notes ‘Analysis Tools’. The explicit conversion of coefficients is given in App. A, p. 21. The surface mass density is represented in the same way,
\[
\sigma(\Omega) = \sum_{lm} \Sigma_{lm} Y_{lm}(\Omega)
\]
(5)
If we assume a linearised theory we can expect a proportionality between excitation and response:
\[
\{[U, V, W, \Phi]_{lm}\} = A\{\Sigma_{lm}\}
\]
(6)
If we assume a radially stratified Earth structure the relation between the load and the response only depends on the distance between load and observation, which means the field quantity describing the observation, \(\phi\), can be written as a convolution integral, \(g_\phi\),
\[
\phi(\Omega) = a^2 \int_{\Omega_0} \sigma(\Omega') g_\phi(\gamma) \, d\Omega',
\]
(7)
where \(\gamma = (|\Omega - \Omega'|)\) is the distance between the two coordinate pairs on the sphere and \(g_\phi\) is the Green’s function for the considered scalar field quantity, \(\phi\). The depends only on \(\gamma\) allows the Green’s functions to be represented by Legendre functions:
\[
g_e(\gamma) := a/M_e \sum_{l} (1 + k_l) P_l(\cos \gamma),
\]
(8)
\[
g_u(\gamma) := a/M_e \sum_{l} h_l P_l(\cos \gamma)
\]
(9)
\(^1\)The representation of vector spherical harmonics is quite compact, for details see Martinec (2000). The full calculus is outlined in Varshalovich \textit{et al.} (1988).
Considering the load in (5), the displacements can be written then

\[ e(\Omega) = \frac{3}{\rho} \sum \frac{1 + k_l}{2l + 1} \Sigma_{lm} Y_{lm}(\Omega) \]  

(10)

\[ u(\Omega) = \frac{3}{\rho} \sum \frac{h_l}{2l + 1} \Sigma_{lm} Y_{lm}(\Omega) \]  

(11)

where \( h_l \) and \( k_l \) are the load Love numbers. If we look at their functional behaviour Figure 1, two things are of interest. The degree 0 does not appear in this figure. This is not due to the logarithmic scale but also due to the fact that it does not appear in the usual listing of Love numbers, e.g. by Farrell (1972). Furthermore \( k_1 = -1 \), this motivates a small excursion.

1.4.1 Excursion to physical meaning of degree 0 and 1

The displacements fields represented by Legendre degree 0 and 1 have to be considered separately. This is due to the integral character of these fields. the surface integral over a scalar spherical harmonics results in

\[ \int_{\Omega} Y_{lm} \, d\Omega = \sqrt{\frac{4\pi}{2l + 1}} \delta_{l0} \delta_{m0} \]  

(12)

For the surface mass density, \( \sigma \) (5), it means a finite mass of the perturbation, which violates the principle of mass conservation (2). So, mass conservation implies that \( \Sigma_{00} = 0 \) and from the linearity of the problem, we can conlude that for the assumed model, there is neither a degree-0 component in displacement (11) nor a degree-0 component in the gravitational potential or displacement (10). From this point of view we don’t have to care about a numerical value for degree 0.
For degree 1, the representation of vector spherical harmonics are of interest. There, the integral

$$\int_\Omega S^{(\lambda)}_{jm} dS = \sqrt{\frac{4 \pi}{3}} \delta_j (2 \delta_{\lambda 1} + \delta_{\lambda -1}) e_m$$

shows only spheroidal components, $\lambda = \pm 1$, in degree $l = 1$. The $e_m$ are the covariant spherical base vectors. Considering the average motion of the surface, (3)

$$u_{\text{CF}} := \frac{1}{A} \int_{\partial V} u dS = \frac{1}{4 \pi} \int_{\Omega_0} \sum_j \sum_m \left[ U_{jm} S_{jm}^{(-1)} + V_{jm} S_{jm}^{(1)} + W_{jm} S_{jm}^{(0)} \right] d\Omega,$$

we get in fully normalised complex spherical harmonics (Klemann & Martinec, 2009)

$$u_{x,\text{CF}} = -\frac{1}{2} \sqrt{\frac{2}{3 \pi}} \text{Re}\{U_{11} + 2 V_{11}\}$$
$$u_{y,\text{CF}} = \frac{1}{2} \sqrt{\frac{2}{3 \pi}} \text{Im}\{U_{11} + 2 V_{11}\}$$
$$u_{z,\text{CF}} = \frac{1}{2} \sqrt{\frac{1}{3 \pi}} (U_{10} + 2 V_{10})$$

That means, that the center-of-figure motion is described by the degree-1 components of the displacement field. Similar, the degree-1 perturbation of the potential describes the center of mass motion and the difference is the so called geocenter motion, which is invariant of the chosen earth related reference frame.

Here, the representation by load-Love numbers, is given, where the surface-mass coefficients are the valid for the $4\pi$-normalised real spherical harmonics, Stokes’ coefficients, and $u$ is in Cartesian coordinates pointing to ($e_x = 0^0\text{N}/0^0\text{E}$), ($e_y = 0^0\text{N}/90^0\text{E}$) and ($e_z = 90^0\text{N}$):

$$u^{\text{gc}} = \frac{1}{3 \rho} \left[ h_1 + 2 l_1 - 3 (1 + k_1) \right] \begin{pmatrix} \Sigma_{11}^C \\ \Sigma_{11}^S \\ \Sigma_{10}^S \end{pmatrix}$$

From Figure 1, we observe that $k_1 = -1$. Not shown is the value $l_1 = 0.113$ of Farrell. This relation allows a quick assessment about the order of the geocenter motion. The specific value of $k_1 = -1$ in Farrell, is due to the chosen reference frame in which the kinematics of the solid earth is described. In Farrell it is the so-called center of the solid earth. For details see Lavallée et al. (2006). Due to GGOS this quantity is of interest and its more precise determination also one of the tasks for the next years. With respect to GIA, there are a number of studies, e.g. Greff-Lefftz (2000); Argus (2007); Klemann & Martinec (2009). As stated above the center of mass motion is described by $1 + k_1$. 

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1.4.2 Toroidal motion

A further result is, that for a surface load that acts as an attracting mass and as a vertical loading pressure on a spherically symmetric earth, a toroidal motion will not be excited, i.e. spheroidal and toroidal motions are decoupled. This is the reason why only three love Numbers describe the deformational behaviour of the earth body in response to a load, $h, l$ for spheroidal displacement and $k$ for gravity. If we assume a laterally varying earth structure this condition is not fulfilled any more and we get a coupling between different degrees and a coupling to the toroidal part (Klemann et al., 2008).

The extension to a time dependent love number which is demanded for viscoelastic behaviour will add a convolution in the time domain.

2 The sea-level equation

The sea-level equation describes the mass redistribution between ice and ocean in a gravitational consistent way. The ocean is considered to follow the geoid. The geoid is calculated from the surface mass redistribution. The surface mass redistribution is considered to deform the solid earth and so, changes the gravity potential. Surface displacement and displacement of gravity potential defines the placement of the ocean which again modifies the load. This complicates the set up of the problem.

2.1 The concept of geoid

The static response of the ocean means that the sea-level follows the geoid which is generated by the mass redistribution. The geoid is here considered in the classical definition:

> The geoid is that equipotential surface which would coincide exactly with the mean ocean surface of the Earth, if the oceans were in equilibrium, at rest (relative to the rotating Earth), and extended through the continents (such as with very narrow canals). According to C.F. Gauss, who first described it, it is the ”mathematical figure of the Earth”, a smooth but highly irregular surface that corresponds not to the actual surface of the Earth’s crust, but to a surface which can only be known through extensive gravitational measurements and calculations.

Wikipedia
This definition deviates from that given in e.g. the IERS convention (Petit & Luzum, 2010), where the geoid is defined by a potential value Groten (2004):

$$W_0 = 62636856.0 \text{m}^2 \text{s}^{-2} \pm 0.5, \text{m}^2 \text{s}^{-2}$$  \hspace{1cm} (17)

This difference can lead to misinterpretations in literature. Whereas it is convenient to calculate this shift by the Bruns’ formula, it will not hold for Gauss definition if a process changes the global sea level. The classical Farrell & Clark (1976) paper for instance use Gauss’ definition. So, a suggestion to the students, take care that you are using the terms correctly and your correspondent does the same.

So, considering a change of the gravity potential due to a dynamic process, the displacement of the potential surface can be calculated from Bruns’ formula,

$$e = \frac{\phi_1}{g_0}$$  \hspace{1cm} (18)

with $g_0$ the normal gravity. This displacement, $e$, should not be mixed with the geoid change, $n$, which also contains a change of the total mass of the ocean or a steric change (thermal expansion). If we represent $n$ and $e$ in spherical harmonics, they will differ in the degree 0 component, which is 0 only for $e$ (25).

### 2.2 Definition of sea level

The sea level was defined uniquely as long the geodesist remained to stay at the shore line and measured its height variations using tide gauges. There it was clear, the sea level was measured relative to the land surface (shore line) at a specified epoch, the relative sea level. This concept also applies to geological or historical markers or indicators of former sea level. With satellite altimetry this view changed, where the sea-level is measured independent from the land surface.

The geoid from Gauss’ definition is

$$n(\Omega, t) = \phi(\Omega, t) + h_{wl}(t)$$  \hspace{1cm} (19)

with $h_{wl}$ the shift between the reference-potential height and the current potential the sea level is following.

This means from a modelling perspective where we can predict field quantities w.r.t. displacement, $u$, and geoid, $n$, in a specified reference frame,

$$h_{RSL}(\Omega, t) = [n - u](\Omega, t) - [n - u](\Omega, t_0)$$  \hspace{1cm} (20)

as the relative sea level and

$$h_{alt}(\Omega, t) = n(\Omega, t) - n(\Omega, t_0)$$  \hspace{1cm} (21)
the altimetric sea level.

Here, one has to keep in mind that \( h_{\text{alt}} \) (21) is not invariant against the considered reference system. The same holds for the prediction of surface motion \( u \). Also here, the translation of the whole network can differ between prediction and observation (Section 3, p. 12).

### 2.3 The ocean function

The ocean function is quite important. It defines the integration domain where the water level is affected by displacement and geoid. In this respect it is a masking function

\[
O(\Omega, t) = \begin{cases} 
0 & \text{if } T(\Omega, t) > 0 \\
1 & \text{if } T(\Omega, t) \leq 0 
\end{cases} \tag{22}
\]

If a deformation process is considered the topography, if defined relative to the mean sea level, may change. Considering an initial state where the topography is specified, \( T_0 = T(t = 0) \), and the perturbation in relative sea level, \( h_{\text{RSL}}(t = 0) = 0 \), the topography defined in the above sense is calculated by

\[
T(\Omega, t) = T_0 - h_{\text{RSL}}(\Omega, t) \tag{23}
\]

This concept is quite important in GIA, where the sea-level not only changes vertically due to uplift but also the horizontal extension of the ocean, i.e. the ocean mask becomes a function of time. This motivates its name 'ocean function'. The currently applied theory is outlined in Kendall et al. (2005).

### 2.4 Moving coast lines

Figure 2 shows an example of how much the sea level varied during the last glacial cycle. At the last glacial maximum the global sea-level was approximately 130 m below its present height, therefore many continental shelves where dry areas. These
regions were unloaded during the LGM time range, which influences of course the dynamic response of the solid earth.

2.5 The coupling between sea-level variations and solid earth

As outlined in Section 1.4, p. 4, the earth responds to a surface load by deformation and so the earth’s surface geometry and the gravity potential will change. The loading consists of the ice load on the continent and the ocean load due to mass redistribution.

\[ m_{\text{load}}(\Omega) = m_{\text{ice}}(\Omega) + m_{\text{oce}}(\Omega). \]  

The mass redistribution implies

\[ \int_{\Omega_0} m_{\text{load}}(\Omega) \, d\Omega = 0 \]  

2.6 Solution of the sea-level equation

The sea-level equation is an integral equation (Farrell & Clark, 1976):

\[ h_{\text{RSL}}(\Omega, t) = [h_{\text{wl}}(t) + e(\Omega, t) - u(\Omega, t)] \mathcal{O}(\Omega, t), \]  

where the homogeneous part describes the shift of the reference geoid

\[ h_{\text{wl}}(t) = \frac{-M_{\text{ice}}(t)}{\rho_{\text{oce}} A_{\text{oce}}(t)} - \frac{1}{A_{\text{oce}}(t)} \int_{\Omega} [e(\Omega, t) - u(\Omega, t)] \mathcal{O}(\Omega, t) \, d\Omega \]  

and

\[ A_{\text{oce}}(t) = \int_{\Omega} \mathcal{O}(\Omega, t) \, d\Omega \]  

It is important to note here, that also for the case of fixed coastlines the equivalent sea level has to be determined by iterations and is not known from the beginning. Considering these aspects we end up with the following solution of the sea-level equation:

\[ (e - u)_i(\Omega) = g_{e-u} \ast [m_{\text{load}}(\Omega) + h_{\text{rsl}}(\Omega) \rho_w \mathcal{O}(\Omega)], \]  

\[ h_{\text{rsl}}(\Omega) = h_{\text{wl}} + (e - u)_i(\Omega) \mathcal{O}(\Omega), \]  

\[ h_{\text{wl}} = \frac{\int_{\Omega} m_{\text{load}}(\Omega)}{\rho_w A_o} - \frac{1}{A_o} \int_{\Omega} (e - u)_i(\Omega) \mathcal{O}(\Omega) \, d\Omega \]  

\[ h_{0}\text{rsl} = \frac{-\int_{\Omega} m_{\text{load}}(\Omega)}{\rho_w A_o}, \]
2.7 Features of its solution

The sea level does not respond uniformly on the melting of ice. Aspects are:

- The solid earth deforms due to changes in loading. Near the reducing load it will move upward.
- The ice load attracts the ocean due to its mass which means, if the ice is melting, the sea level will drop in the areas around the ice load.
- In the farfield the sea level rises due to the additional amount of water.

3 Reference systems

A unique definition of a reference system is not possible. In fact, the earth is wobbling around the sun and wobbling itself, which makes it difficult for a person on the earth’s surface to define the motion of a distributed range of points on the earth surface in an invariant reference frame. This is important in order to predict, from the set of motions by a functional relation i.e. physical process, the motion at other points of the surface. These reference frames are usually defined by the average motion of the set of points w.r.t. a fixed coordinate system at each epoch. In secular trends one important aspect is the stability of reference systems. But also from the numerical modelling point of view, a reference system has to be prescribed. Sometimes, this is implicitly assumed in the field equations, sometimes it has to be done explicitly. For a body in space we have to prescribe (or fix) 6 components: 3 describing the position and motion in space and 3 describing its orientation. For the latter, in geodesy usually no net rotation of the surface is assumed but other definitions are also possible:

1. no surface net rotation (NF) $\int_{B} \rho \mathbf{e}_r \times \mathbf{u} \, dS = 0$
2. conservation of total momentum (NM) $\int_{B} \rho \mathbf{e}_r \times \mathbf{u} \, dV + \int_{\partial B} \mathbf{e}_r \times \sigma \, dS = 0$
3. no lithosphere net rotation (NL) $\int_{B_L} \rho \mathbf{e}_r \times \mathbf{u} \, dV = 0$
4. no internal rotation (NE) $\int_{B} \rho \mathbf{e}_r \times \mathbf{u} \, dV = 0$
5. no mantle rotation (NMa) $\int_{B_M} \rho \mathbf{e}_r \times \mathbf{u} \, dV = 0$

The position/motion in space is specified in IRTF2005 as the centre of mass, previously it was the centre of figure, the Love numbers are referenced to the center of internal masses. So, also here one has to be careful especially if discussing global processes, where derived motions can be depend on the considered reference frame:

1. centre of mass (CM) $\int_{B} \rho \mathbf{r} \, dV + \int_{\partial B} \sigma \mathbf{r} \, dS = 0$
2. centre of figure (CF) \( \int_{\partial B} \mathbf{u} \, dS = 0 \)

3. centre of deformation (CD) \( \int_B \mathbf{u} \, dV = 0 \)

4. centre of internal masses (CE) \( \int_B \rho \mathbf{r} \, dV = 0 \)

The advantage of CM is that it describes the center of satellite motions. The advantage of CF is its realisation for a network of GPS stations.

A definition which is invariant to the latter conventions is the geocenter motion

\[
\mathbf{u}_{gc} := \mathbf{u}^{CF} - \mathbf{u}^{CM}
\]  

(33)

This reduction to degree 1 of the displacement field can be explained by the definition of the centre of figure and centre of mass motion,

\[
\mathbf{u}^{CF} := \frac{1}{\Omega} \int_{\Omega} \mathbf{u} \, dS
\]  

(34)

The observed GC is dominated by a seasonal signal (e.g. Rietbroek et al., 2011). At the moment, the accuracy of GC motion is not better than 1 mm/yr. The secular signal is about 1 mm/yr, i.e. 0.1 mm/yr is the demanded accuracy to analysis this kind of signal.
4 Glacial isostatic adjustment

In literature there are two phrases commonly used to denote the process we will discuss during this course:

GIA glacial isostatic adjustment

PGR post glacial rebound

There, we can extract following competitive terms

**glacial** means related to glaciers and/or glacial cycle,

**isostatic adjustment** the movement to a new equilibrium state of forces,

**post glacial** means after end of the glaciation period,

**rebound** movement due to a disequilibrium.

Both definitions imply that there exists a static equilibrium state which is reached through time. The principle of this process is shown in Figure 3, where the arrows show the strain inside the lithosphere and the flow inside the mantle, respectively. To understand the different meaning of the two expressions, we have to discuss what glacial stands for. As part of ‘glaciology’ it is related to the scientific discipline dealing with ice at the earth’s surface and as part of glaciation it means the geological time intervals during which large parts of the earth’s surface were covered by ice. Therefore, PGR describes the dynamic process after the last glaciation period and GIA means the response of the solid earth to any ice load redistribution.

Due to mass conservation, the ice-load variations have to be considered together with variations in sea level, which results in the modern definition of glacial isostatic
adjustment: GIA describes the ongoing adjustment of the earth’s interior to surface loading that is attributed to the changing mass distribution of ice and water.

The dimensions we have to deal with can be summarised as follows:

- extension of ice sheets $\mathcal{O}(1000\text{ km})$
- thickness of ice sheets $\mathcal{O}(1000\text{ m})$
- duration of process $\mathcal{O}(10,000\text{ yr})$
- termination of main glaciation 8,000 yr bp
- motion in previously glaciated regions of Scandinavia and Canada $\mathcal{O}(1\text{ cm/yr})$

So, we have to answer:

*How deform glacial loads the earth?* – ‘Rebound’ (germ.: Rückfederung, Erholung) implies already the understanding of the lithosphere as an elastic plate. This concept from plate tectonics can be used to describe the flexural behaviour of the lithosphere in response to loading.

Elastic plate is defined by technical mechanics as a thin plate, symmetric stress pattern, ... and its strength is simply the flexural rigidity. In geophysics, this fills already books (e.g. Watts, 2001).

The loads considered in GIA are glaciers or ice sheets which covered large areas of the continents on the northern hemisphere as additional masses. These ice sheets have to be carried by the lithosphere and the mantle below. On the time scale of glaciation process, the mantle does not react as an elastic body but as a viscous fluid. In consequence, we assume the lithosphere to be floating on the mantle, and we get the first physical principle which is the equilibrium of momentum:

\[
\text{Loading force} = \text{flexure of lithosphere} + \text{buoyancy (isostasy)}
\]

The process is slow, so we neglect the inertial forces. The buoyancy describes the fact, that the load deforms the lithosphere which is floating on the mantle.

\[
D \frac{d^4 z}{dx^4} + \rho_m g z = \rho_{\text{ice}} g h
\]

where

\[
D = \frac{E T^3}{12(1-\nu^2)}
\]

This is, the flexure of a beam with flexural rigidity, $D$. The buoyancy of the underlying material with density, $\rho_m$, is in equilibrium to the surface load. Of course, $z$ is vertically down and $x$ the horizontal of this 2d problem. From the equilibrium condition (35), it is evident that this equation describes the static state due to the fact that the mantle material is assumed to be an inviscid fluid.
The most important effect which makes this process such interesting is missing in both phrases (GIA and PGR), which is the retardation response, – the reason why still a vertical motion is observed. The mantle is not an ideal but a viscous fluid. Therefore, the adjustment or rebound due to the last glacial cycle is an ongoing process.

So, we have to consider two physical phenomena, in order to describe the process:

- elastodynamics, which we need to describe mathematically the flexure of the lithosphere, and
- fluid dynamics to describe the viscous flow inside the mantle.

Both disciplines are part of the continuum mechanics and there, the two end members of the process:

\[ \text{strain} \propto \text{stress: } \mu \varepsilon = \tau \]
\[ \text{strain rate (flow) } \propto \text{stress: } \eta \dot{\varepsilon} = \tau \]

The third process which is present, is gravity, because buoyancy \( \propto \) gravity: \( b = \nabla (\rho_m g \cdot u) \)

One interesting aspect to note is that the viscosity of the earth’s interior can only be quantified by a dynamic process like GIA. Therefore, GIA is the discipline which gave the first estimates about the viscosity of the earth (e.g. Haskell, 1935), which is \( 10^{21} \) Pa s.

As stated above to consider the process of GIA we have to consider elastic and viscous behaviour at the same time, especially because although GIA is long time process it mainly describes a status of disequilibrium. Therefore we are interested in a formulation of the rheology where elastic and viscous behaviour are described at the same time. This is possible by viscoelasticity. The most elementary relation is the Maxwell body,

\[ \dot{\varepsilon} = \tau/\mu + \tau/\eta \]  \hspace{1cm} (37)

This material law is considered generally in GIA. The main parameter to be investigated is the dynamic viscosity, \( \eta \), whereas the shear modulus is considered from standard earth models like the Preliminary Reference Earth Model (PREM) from Dziewonski & Anderson (1981).

### 4.1 Field equations describing the solid earth response

The field equations here cited from Martinec (2000) describe the displacements of a viscoelastic, incompressible, non-rotating, self-gravitating continuum in a spherical geometry. They consist of the equation of motion

\[ \nabla \cdot \tau - \rho_0 \nabla \phi_1 + \nabla \cdot (\rho_0 u \nabla \phi_0 - \nabla (\rho_0 u \cdot \nabla \phi_0) = 0 , \]  \hspace{1cm} (38)
the potential equation
\[ \nabla^2 \phi_1 + 4 \pi G \nabla \cdot (\rho_0 u) = 0, \]
(39)

the constitutive equation of Maxwell viscoelasticity, here in 3d,
\[ \dot{\tau} = \dot{\tau}^E - \frac{\mu}{\eta} (\tau - \Pi I) \quad \tau^E = \Pi I + 2 \mu \epsilon \]
(40)

and the continuity equation, here formulated for the case of incompressibility.
\[ \nabla \cdot u = 0. \]
(41)

These field equations have to be solved inside the solution domain, \( \mathcal{B} \), for displacement \( u \), stress \( \tau \) and the potential \( \phi_1 \). The material parameters considered, are the density, \( \rho \), the shear modulus, \( \mu \), and the viscosity, \( \eta \). The solution domain usually ranges inside a spherical shell from the surface to the core mantle boundary. The principles of this theory are outlined in Tromp & Mitrovica (1999).

The boundary conditions are at the surface the absence of any traction and free displacement and at the core-mantle boundary the conditions which describe the coupling to a homogeneous fluid sphere. The excitation is then represented by a surface load an internal load or a potential perturbation.

### 4.2 Solution of field equations

The classical method for solving the field equations is to transfer the time dependence which only appears in (40) into the spectral domain. This is done by Laplace transformation. Then the Laplace transformed equation of motion corresponds to the elastic problem, only that the shear modulus becomes a function of \( s \). The horizontal dependence of the solutions are represented by scalar-, vector- and tensor-spherical harmonics, e.g. (3) and (4). The differential equation remains only with respect to radial distance, \( r \). The system of equations can be represented for homogeneous layers by a first-order \( 6 \times 6 \)-differential system,
\[ \frac{d}{dr} Y_l(r) = A_l(r) Y_l(r) \]
(42)

which can be solved then for a stratified continuum by propagator matrices. As discussed in Section 1.4, p. 4 in the case of spherical symmetry, \( Y \) contains \( U, V \) representing the displacement the potential perturbation and respective terms of their first derivatives. The solution of this homogenous system of first order differential equations can be represented by its eigen modes, the so called relaxation time spectrum. For the specific form of \( A \), which depends also on the Laplace parameter, \( s \), the solution can be represented by \( Y^S / \det M(s) \). The determinant of the fundamental matrix \( M \) is unique for \( A \) and does not depend on the excitation.
represented by boundary conditions. The transformation back into the time do-
main, demands an inverse Laplace transformation. Here, the usual way is to apply
the residue theorem, by identifying the roots of \( \det M(s) \), the eigenmodes of the
system. Then, the solution is represented by the sum of the eigenmodes with an
appropriate weighting. This method is based on Peltier (1974, 1976) and widely
used (Sabadini & Vermeersen, 2004).

Martinec (2000) formulated the equations as an initial-value problem and the time
dependence is solved directly in the time domain. This has a number of advantages:
(1) the earth model can be coupled with a dynamic ice model. (2) It is possible to
consider also lateral variations of viscous parameters in the earth structure. (3) In
addition to a linear rheology also stress dependent rheologies can be considered.

5 Further reading


Herring (2009): Geodesy – A nice new book about what I am not so familiar
with.

of the Solid Earth – My state of the art compenion discussing the rheological
aspects of the earth’s interior. It replaces the older standard by Ranalli (1987).

Kendall et al. (2005): article – Outline of the sea-level equation applied in GIA.

perspective.

about application of normal mode theory for a viscoelastic planet.

Varshalovich et al. (1988): Quantum Theory of Angular Momentum – If you re-
ally have to work with spherical harmonics.

Watts (2001): Isostasy and Flexure of the Lithosphere – For me, it is a standard
when getting information about mechanical aspects of the lithosphere.

Whitehouse (2009): Glacial isostatic adjustment and sea-level change: State of
the art report – A rather new summary about GIA and sea-level equation.
Easily accessible from www.skb.se.
References


A Conversion of Stokes’ coefficients

According to Pěč & Martinec (1982), the relation between coefficients related to real $4\pi$-normalised spherical harmonics, Stokes’ coefficients, and the fully normalized complex spherical harmonics are

\begin{align}
A_{j0} &= \sqrt{4\pi} \bar{C}_{j0} \\
A_{jm} &= (-1)^m \sqrt{2\pi} (\bar{C}_{jm} - i \bar{S}_{jm}), \quad m > 0 \\
A_{j-m} &= (-1)^m A_{jm}, \quad m > 0
\end{align}

where $[C, S]_{jm}$ are the Stokes’ coefficients and $A_{jm}$ are the coefficients of complex spherical harmonics.

B Vector spherical harmonics

Based on the complex normalized spherical harmonics with Condon-Shortly phase, it is straightforward to define vector spherical harmonics.

\begin{align}
S_{jm}^{(-1)} &= Y_{jm} e_r \\
S_{jm}^{(1)} &= \nabla_\Omega Y_{jm} \\
S_{jm}^{(0)} &= (e_r \times \nabla_\Omega) Y_{jm}
\end{align}

From these a number of integral relations can be derived. The most prominent in this respect is

$$
\int_\Omega S_{jm}^{(\lambda)} dS = \sqrt{\frac{4\pi}{3}} \delta_{j1} (2 \delta_{\lambda1} + \delta_{\lambda-1}) e_m
$$

which shows that the average motion of a surface is only expressed by spheroidal ($\lambda = \pm 1$) and degree-1 ($l = 1$) components. The $e_m$ are the covariant spherical base vectors.
In der Schriftenreihe des Instituts für Geodäsie und Geoinformation der Rheinischen Friedrich-Wilhelms-Universität Bonn sind erschienen:

Heft 30  Annette Eicker / Jürgen Kusche (eds.)
2013  Lecture Notes from the Summer School of DFG SPP1257 Global Water Cycle

Heft 29  Matthias Siemes
2012  Ein Beitrag zur koordinatengesteuerten Aussaat von Rübenpflanzen mittels Multi-Sensor-System und Filteransatz

Heft 28  Jörg Schmittwilken
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Ein Beitrag zur Entwicklung des Liegenschaftskatasters im Lande Nordrhein-Westfalen in Vergangenheit, Gegenwart und Zukunft

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Heft 25  Benedikt Frielinghaus
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2011  Entwicklung eines Kalman-Filters zur Bestimmung kurzzeitiger Variationen des Erdschwerefeldes aus Daten der Satellitenmission GRACE

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Heft 22  20th Meeting of the European VLBI Group for Geodesy and Astronomy
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2007  Grouping Uncertain Oriented Projective Geometric Entities

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2007  Geodäsie und Quantenphysik
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